Intended for the Matriculation Students of the Indian Universities.

ALGEBRAICAL
EXERCISES WITH SOLUTIONS.

CONTAINING NEW PARAGRAPHS ELUCIDATING PRINCIPLES OF THE REMAINDER THEOREM, NOTATION OF FUNCTIONS, GRAPHS AND QUADRATIC EQUATIONS, WITH ILLUSTRATIVE EXAMPLES OF DISTINCT TYPES FOLLOWED BY EASY SOLUTIONS.

BY

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PREFACE.

This manual was originally compiled with a view to place within easy reach of students preparing for the Matriculation Examination of the Indian Universities a collection, on a limited scale, of typical examples in Algebra—such as are indispensably present before the mind of experienced teachers and examiners alike. They have been selected from approved standard works, as well as from various examination papers, and grouped into exercises as model questions arranged in progressive order. Solutions follow each exercise, not with the object of fostering what in University parlance goes by the name of "Cram," but to engender in the minds of tiros that love for neat and short methods, without which no taste for mathematics or science in general can be acquired In working out each example, therefore, particular care has been taken to get at the answer by the shortest route, and all tedious and round-about processes have been avoided as much as possible It is, however, known to all engaged in teaching mathematics, that pupils of average intelligence at first often fail to see the reasons for the steps in concise and compact operations. To obviate this inconvenience—the only objection to short methods— ach step has been elucidated by indicating the principles bearing upon it —principles, which, for the purpose of constant reference, hav been embodied with explanations, wherever necessary, in twenty-nine Articles in the Introductory chapter.

After having gone through two editions which were well received by both teachers and students, the book lay out of print for nearly two decades owing to the auth

having lost interest in it under stress of several sad circumstances. At the insistence of many of his old pupils who found the book useful in their school days and who are now engaged in educational work, a revised and considerably enlarged edition is brought out It is needless to say that it is placed abreast of the times meeting, as it does, the requirements of the up-to date syllabus of the Indian Universities It is hoped that this little hand-book will prove as useful and instructive to the present generation of students as it did to their predecessors.

The services rendered by Babu Pransankar Sen M.A B.L., Vakil, (of 21 Scott's Lane Calcutta) in passing the manuscript through the press and in identifying himself with the publication of this new edition of the book deserve to be mentioned with warmest gratitude.

'Biram Geha' (বিরাম গেহ)
27, Nawapura Benares City
7th February 1916.

S. C. M.

ADDENDA and CORRIGENDA.

Students are particularly requested to supply omissions and note corrections in their proper places before commencing study of the book.

I OMISSIONS.

Page 8 Add another note after the 2nd line in a new paragraph as follows :—

If d be changed into $(-d)$, $x-d$ becomes $x-(-d)$ which $=x+d$ (Art 3.)

If the given expression (dividend) be divided by $x+d$, the remainder will be obtained by changing d into $(-d)$ throughout in R, that is to say $R=f(-d)=(-d)^n+a.(-d)^{n-1}+b.(-d)^{n-2}+c(-d)^{n-3}+\cdots\cdots+f$

Illustration Divide $7x^6+3x^5+5x^4+2x^3+4x^2+x+10$ by $x+3$
$R=7.(-3)^6+3(-3)^5+5(-3)^4+2.(-3)^3+4(-3)^2+(-3)+10$
$=7.729-3.243+5.81-2.27+4.9-1.3+10=5554-786=4768.$

Page 54 After *Note* 1 write *Note* 2 which has been misplaced on page 55 line 3

II. CORRECTION OF MISPRINTS

Page 8, line 3, change n *into* m.

Page 14, Fourth line of Art 20, change β into b

Page 15, line 7, change X′ into X (without accent) ; line 13, change Y into Y′ (with accent)

Page 38, Example 4. (b) change $ax-a$ into $xa-a^2$; Page 39 line 14 *sheep* for *shep*

Page 40. Top, write 7 opposite the Graph-diagram indicating Answer to the 7th question.

Page 51. line 17, Write (at the end of the line), $=0$

Page 56, line 1, change the thick type **3** into the numeral 3 (common type).

Page 58, sixth question 1st line. Write "field" after "rectangular".

Page 59, In the 3rd line of the small type Note, place a comma after the bracket.

Page 70, line 3, change $y^{\frac{x}{x}}$ into y^{x}.

Page 72, line 3 from the bottom change $+3pq^2$ into $-3pq^2$.

Page 98. Example 4 (a) second line change the figure 6 into the letter b.

Page 120, Example 6, second line, z should be z^2.

ALGEBRAICAL

EXERCISES WITH SOLUTIONS.

INTRODUCTION.

[*** *Students are recommended to carefully get up the following articles, to which reference will frequently be made in the solutions.*]

1. If $a=2$, $b=3$,

$ab=2\times 3$ and not $=23$.

If it be required to express 23 by means of the above letters a and b, we shall do so by $10a+b$

Similarly if $c=4$.

234 is expressed by $100a+10b+c$; so again if $d=5$, 2345 is expressed by $1000a+100b+10c+d$.

2. $a+b\times c$ is different from $(a+b)\times c$, in as much as the former means, a is added to the product of b and c; while the latter means that the sum of a and b is multiplied by c.

Hence, the terms enclosed in a bracket are to be considered as *one* term.

3. $a-(b+c)=a-b-c$.

$a-(b-c)=a-b+c$.

Hence the rule that if the sign − (minus) precedes a bracket, which encloses several terms, the signs of the terms enclosed must be changed on the removal of the bracket.

Note.—If the sign + (plus) precedes a bracket, the bracket can be removed at pleasure, provided there be no other purpose to be served by it For example, we cannot remove the bracket from $a+(b+c)x$ without affecting its value for here the bracket means that x is to be multiplied by the sum of b and c. But if we simply took off the bracket, the result would mean that x was to be multiplied by c only.

4. $a^m \times a^n = a^{m+n}$. Illustration : $x^2 \times x^3 = x^{2+3}$ or x^5.

Conversely, $a^{m+n} = a^m \times a^n$.

Illustration. $a^5 = a^{3+2} = a^3 \times a^2$ or $a^{4+1} = a^1 \times a^4$.

5. $(a^m)^n = a^{mn}$. Illustration : $(x^2)^3 = x^{2\times 3} = x^6$.

Conversely, $a^{mn} = (a^m)^n$.

Illustration. $a^6 = a^{3\times 2} = (a^3)^2$ or $= (a^2)^3$.

6. $a^m \div a^n = a^{m-n}$. Illustration. $x^5 \div x^3 = x^{5-3} = x^2$.

Conversely, $a^{m-n} = a^m \div a^n$.

Illustration : a^2 or $a^{5-3} = a^5 \div a^3$.

7. $\sqrt[n]{a^m} = a^{\frac{m}{n}}$. Illustration. $\sqrt[2]{x^3} = x^{\frac{3}{2}}$.

Conversely, $a^{\frac{m}{n}} = \sqrt[n]{a^m}$. Illustration. $a^{\frac{3}{2}} = \sqrt[2]{a^3}$.

8. $a^{-m} = \dfrac{1}{a^m}$.

For, $-m = m - 2m$

$\therefore \quad a^{-m} = a^{m-2m} = a^m \div a^{2m}$. [Art. 6. Conv.

$$= \frac{a^m}{a^{2m}} = \frac{a^m}{a^{m+m}} = \frac{a^m}{a^m \times a^m}$$ [Art. 4. Conv.

$$= \frac{1}{a^m}.$$

Illustration : $a^{-2} = \dfrac{1}{a^2}$; and $a^{-3} = \dfrac{1}{a^3}$

Conversely, $\frac{1}{a^m}=a^{-m}$, Illustration. $\frac{1}{a^5}=a^{-5}$.

Note. $a^0=1$.

For $a^0=a^{m-m}=a^m \div a^m$ [Art. 6. Conv.

$=1$.

i.e. any number raised to the power $0=1$.

9. $(+a)\times(+b)=+ab$.

$(+a)\times(-b)=-ab$.

$(-a)\times(-b)=+ab$.

Or like signs produce +, and unlike signs produce −, in performing operations of multiplication and division. Therefore $(+a)^m=+a^m$ whether m is odd or even, and $(-a)^m=-a^m$ when m is odd, and $+a^m$ when m is even.

10. The formulæ for squaring.

A. $(a+b)^2=a^2+b^2+2ab$; and not $=a^2+b^2$.

B. $(a-b)^2=a^2+b^2-2ab$; and not $=a^2-b^2$.

C. $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$ and not $=a^2+b^2+c^2$.

Note.—If we write $-a$ in the form $+(-a)$, we can arrive at important results ; for example, from the formula **C** we can square $a+b-c$ by giving to $-c$ the form $+(-c)$ Thus :—

$(a+b-c)^2=\{a+b+(-c)\}^2$

$=a^2+b^2+(-c)^2+2ab+2a\times(-c)+2b\times(-c)$

$=a^2+b^2+c^2+2ab-2ac-2bc$ [Art 9.

So again, $(a-b-c)^2=\{a+(-b)+(-c)\}^2$

$=a^2+(-b)^2+(-c)^2+2a\times(-b)+2a\times(-c)$
$+2\times(-b)\times(-c)$

$=a^2+b^2+c^2-2ab-2ac+2bc$. [Art. 9.

Note.—We can express the result of the formula **C** in general terms thus :—

The square of any expression consisting of several terms is equal to the sum of the squares of the several terms and twice the product of every two terms taken separately.

For example.

D. $(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$

By writing $-c$ as $+(-c)$ and $-d$ as $+(-d)$,

$(a+b-c-d)^2=a^2+b^2+(-c)^2+(-d)^2+2ab+2a\times(-c)+2a\times(-d)+2b\times(-c)+2b\times(-d)+2\times(-c)\times(-d)$

$= a^2+b^2+c^2+d^2+2ab-2ac-2ad-2bc-2bd+2cd$ [Art 9

11. The formulæ for cubing.

A. $(a+b)^3=a^3+b^3+3ab(a+b)$, and not a^3+b^3.

B. By taking $-b=+(-b)$

$(a-b)^3=\{a+(-b)\}^3$

$=a^3+(-b)^3+3a\times(-b)\{a+(-b)\}$

$=a^3-b^3-3ab(a-b)$, and not a^3-b^3.

12. $(a+b)(a-b)=a^2-b^2$.

Note. This very useful formula is remembered in the following general form :—

The product of the sum and the difference of any two terms is equal to the difference of the squares of these two terms.

13. Formulæ for the product of two binomials having the same first term

A. $(x+a)(x+b)=x^2+x(a+b)+ab$

B. $(x+a)(x-b)=x^2+x(a-b)-ab.$

C. $(x-a)(x+b)=x^2+x(b-a)-ab$

D. $(x-a)(x-b)=x^2-(a+b)x+ab.$

Note.—The results B, C and D are all obtainable from A by writing $-a$ as $+(-a)$ and $-b$ as $+(-b)$.

14. A. $(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc.$

B. $(x+a)(x-b)(x+c)=x^3+(a-b+c)x^2+(ac-ab-bc)x-abc.$

Note.—This result is obtained from A by writing $-b=+(-b)$

15. $(x+a)(x+b)(x+c)(x+d)=x^4+(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2+(abc+abd+bcd+acd)x+abcd.$

Note.—By means of this formula we can multiply any four binomial quantities of which the first terms are the same, only writing as before $-b=+(-b)$.

16 A. $a^2+2ab+b^2=(a+b)^2$ [Converse of 10 A.

Hence, $a^4+2a^2b^2+b^4$
$=(a^2)^2+2a^2b^2+(b^2)^2=(a^2+b^2)^2.$

B. $a^2-2ab+b^2=(a-b)^2$ [Conv. 10 B.

C. $a^2-b^2=(a+b)(a-b)$. [Conv. Art. 12.

Hence, $(x+y)^2-z^2=(x+y+z)(x+y-z)$

17. $ax+bx=x(a+b)$.

Hence, $x(a+b)+y(a+b)=(a+b)\{x+y\}$

or, $x(a-b)-y(a-b)=(a-b)\{x-y\}$

Similarly, $(x+m)(a+b)+(y+n)(a+b)$

$=(a+b)\{(x+m)+(y+n)\}$

$=(a+b)(x+m+y+n)$ [Art. 3. note.

So again, $(x+m)(a+b)-(y+n)(a+b)-(z+p)(a+b)$

$=(a+b)\{(x+m)-(y+n)-(z+p)\}$

$=(a+b)\{x+m-y-n-z-p\}$ [Art. 3.

18. A $a^3+b^3=(a+b)(a^2-ab+b^2)$

For, $a^3+b^3=a^3+a^2b-a^2b-ab^2+ab^2+b^3$

$=a^2(a+b)-ab(a+b)+b^2(a+b)$ [Art. 17.

$=(a+b)(a^2-ab+b^2)$ [Art. 17.

B. $a^3-b^3=(a-b)(a^2+ab+b^2)$

For, $a^3-b^3=a^3-a^2b+a^2b-ab^2+ab^2-b^3.$

$=a^2(a-b)+ab(a-b)+b^2(a-b)$ [Art. 17.

$=(a-b)(a^2+ab+b^2)$ [Art. 17.

C. $a^4 - b^4 = (a-b)\ (a^3 + a^2b + ab^2 + b^3)$

For $a^4 - b^4 = a^4 - a^3b + a^3b - a^2b^2 + a^2b^2 - ab^3 + ab^3 - b^4$

$= a^3(a-b) + a^2b(a-b) + ab^2(a-b) + b^3(a-b)$

$= (a^3 + a^2b + ab^2 + b^3)\ (a-b)$ [Art. 17.

D. $a^5 + b^5 = (a+b)\ (a^4 - a^3b + a^2b^2 - ab^3 + b^4)$

For, $a^5 + b^5 = a^5 + a^4b - a^4b - a^3b^2 + a^3b^2 + a^2b^3 - a^2b^3 - ab^4 + ab^4 + b^5.$

$= a^4(a+b) - a^3b(a+b) + a^2b^2(a+b) - ab^3(a+b) + b^4(a+b)$ [Art. 17.

$= (a+b)\ (a^4 - a^3b + a^2b^2 - ab^3 + b^4)$ [Art. 17.

19. Function.

An expression whose value is liable to change on account of its containing a quantity of constant or fixed value (as numerical figures or letters a, b, c &c having determinate character) and another which is not so fixed *i.e.*, variable (as the letters x, y, z whose magnitude may be anything) is said to be a function of this *variable.*

Thus $2x$, $3x^2$, $\frac{1}{1+x}$, 4^x, $2x^3 + 3x + 1$, ax^n, $bx\ (c + dx^3)$ are functions of x. As such functions are generally long in form, the length is conventionally expressed by the curt form $f(x)$ or $\varphi(x)$ in which form the letters f or φ is not to be separated from (x) but must go hand in hand with (x) *as if the two formed one letter*. It is a mere notation of a lengthened form. Thus if any of the above expressions *e. g.* $bx(c + dx^3)$ be represented by the single letter y we shall say $y = f(x)$.

Illustration : Consider the function $y = x^2(1 - x)$.

Then, when $x=0$ $y=0$

" $x=1$ $y=0$

" $x=2$ $y=4(1-2)=-4$

" $x=3$ $y=9(1-3)=-18$

and so on.

This explains how the value of y depends on the value of the variable x or the meaning of $y=f(x)$

Thus if $f(x)=x^2+x+1$.

$f(0)=0^2+0+1=1$.

$f(1)=1^2+1+1=3$.

$f(2)=2^2+2+1=7$, and so on.

Thus, generally $f(a)=a^2+a+1$.

20. Remainder Theorem.

If a rational integral algebraical expression of the form $x^n+ax^{n-1}+bx^{n-2}+cx^{n-3}+\ldots\ldots+f$......(A).

be divided by $x-d$ until a remainder is obtained which does not involve x, then this remainder will be $d^n+ad^{n-1}+bd^{n-2}+cd^{n-3}+\ldots\ldots+f$.......(B).

Let $f(x)=$ the given expression.

Divide this expression by $x-d$; and let Q be the quotient and R the remainder not containing x.

Then $f(x)=Q\,(x-d)+R$............(C).

It is required to prove that

$R=d^n+ad^{n-1}+bd^{n-2}+cd^{n-3}+\ldots\ldots+f$.

Since R does not contain x, whatever be the value of x, R will *always* be the same.

For our purpose, let us put $x=d$ in (C)

Then $f(d)=Q(d-d)+R$.

$=Q\times 0+R$

$=R$.

therefore $R=f(d)=d^n+ad^{n-1}+bd^{n-2}+cd^{n-3}+\ldots+f$. [Art 19.

Note. This remainder is the same as (A), only x has been changed into d.

Corollary 1. $a^m + b^m$ is divisible by $a+b$ when n is an odd number, but not when m is even number.

For, by remainder theorem, when $a^m + b^m$ is divided by $(a+b)$, the remainder is obtained by putting $a = -b$ in $a^m + b^m$ [This is made obvious by writing $a+b$ as $\{a-(-b)\}$. [Art 3

That is, the remainder is R where $R = (-b)^m + b^m$.

Case 1. m odd.

$$R = (-b)^m + b^m$$
$$= -b^m + b^m \qquad \text{[Art. 9.}$$
$$= 0.$$

Showing that $a^m + b^m$ is divisible by $a+b$.

N. B. The quotient or the other factor is obtained from Art 18

Case 2. m even

$$R = (-b)^m + b^m$$
$$= b^m + b^m \qquad \text{[Art. 9}$$
$$= 2b^m$$

Showing that $a^m + b^m$ is not divisible by $a+b$.

Corollary 2. $a^m - b^m$ is divisible by $a-b$ when m is an integer, be it odd or even.

For, by remainder theorem, the remainder

$$R = (+b)^m - b^m = b^m - b^m = 0 \qquad \text{[Art. 9.}$$

N. B. The quotient or the other factor is obtained, as in the previous case, from article 18.

21. A. If $x + a = b$, $x = b - a$.

For, if equals be taken from equals, the remainders are equaal. [Axiom.

Therefore, taking a from both sides, we get

$$x + a - a = b - a, \quad i.e., \; x = b - a.$$

Similarly.

B If $x-a=b$, $x=b+a$.

For, if equals be added to equals, the wholes are equal.

$\therefore$ adding a to both sides, we get [Axiom.

$x-a+a=b+a$ or $x=b+a$.

Similarly,

C. From the axiom that if equals be multiplied by equals, the products are equal, the following is deduced,

If $\frac{x}{a}=b$, $x=ab$.

D If equals be divided by equals the quotients are equal.

$\therefore$ if $ax=b$, $x=\frac{b}{a}$.

E. If $\pm x=a$, [i.e., or if $+x=a$, if $-x=a$] $x^2=a^2$.

For by multiplying both sides by $\pm x$,

$(\pm x)\times(\pm x)=a\times(\pm x)$, or, $x^2=\pm ax$, but $\pm x=a$ [Hyp.

$\therefore x^2=a\times a=a^2$

Conversely, If $x^2=a^2$, $\pm x=a$, which is the same thing as $x=\pm a$.

Note. The question arises why $\pm$ is not prefixed to x, but is prefixed only to a. The answer is that it is superfluous: for, if we did so, we would get $\pm x=\pm a$ i.e.,

$+x=+a\cdots(1)$ (3) is identical with (2)

$+x=-a\ldots(2)$ (4) " " " (1)

$-x=+a$ (3) . only (1) and (2) are the only

$-x=-a$.(4) necessary values of $+x$.

22. A If $\frac{a}{b}=\frac{c}{d}$ 1 then $\frac{b}{a}=\frac{d}{c}$

for $1=1$ ......

Divide **2** by **1**.

$\therefore 1\div\frac{a}{b}=1\div\frac{c}{d}$ Art 21. D.

or $\frac{b}{a} = \frac{d}{c}$.

This principle of proportion is called *Invertendo.*

B. If $\frac{a}{b} = \frac{c}{d}$ ...... 1 then $\frac{a}{c} = \frac{b}{d}$.

for, multiplying both sides of 1 by $\frac{b}{c}$,

$$\frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}$$ [Art. 21. C.

or $\frac{a}{c} = \frac{b}{d}$.

This principle of proportion is called *Alternando.*

C. If $\frac{a}{b} = \frac{c}{d}$ ...... 1 then $\frac{a+b}{b} = \frac{c+d}{d}$.

for, adding unity to both sides of 1.

$$\frac{a}{b} + 1 = \frac{c}{d} + 1$$ [21. B. Axiom.

or $\frac{a+b}{b} = \frac{c+d}{d}$.

This principle of proportion is called *Componendo*

D. Similarly, subtracting unity from 1

$$\frac{a-b}{b} = \frac{c-d}{d}$$ [21. A. Axiom.

This principle is called *Dividendo.*

E. Dividing C by D,

$$\frac{a+b}{b} \div \frac{a-b}{b} = \frac{c+d}{d} \div \frac{c-d}{d}.$$ [Art. 21. D.

$$\therefore \frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

This combination of *Componendo* and *Dividendo* should be carefully remembered.

23. Quadratic equation.

[Definition : When an equation contains the square and no higher power of the unknown quantity, it is called a quadratic equation or an equation of the second degree]

A. The most general form of the equation of second degree is

$ax^2+bx+c=0$ [Containing both the second and the first power of the unknown x]

All quadratic equations can be cast into this form. Therefore it is enough to solve it.

To do this we proceed as follows :

Dividing both sides by a

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$ [Art. 21. D.

Subtracting $\frac{c}{a}$ from both sides.

$$x^2+\frac{b}{a}x=-\frac{c}{a}.$$ [Art. 21. A.

Adding $\frac{b^2}{4a^2}$ to both sides *i.e.* square of half the coefft. of x

$$x^2+2.\frac{b}{2a}.x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}.$$ [Art 21. B.

$$\text{or } \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$ [Art. 16. A.

$$\text{or } x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$ [Art. 21. E. conv.

$$\text{or } x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$ [Art 21. A.

The values of x satisfying the equation are called the *roots* of the given equation.

B. If the equation reduces to the form of factors as $(x+a)(x+b)=0$, then the solutions are $x=-a$, $x=-b$, for each factor is equal to zero.

Similarly, if $(x-a)(x-b)=0$ $x=a$, $x=b$

So again if $(2x+a)(2x-b)=0$

$$2x=-a \text{ or } b$$

$$i.e.\ x=-\frac{a}{2} \text{ or } +\frac{b}{2} \text{ and so on.}$$

24. To prove that a quadratic equation cannot have more than two roots.

If possible, let the equation

$ax^2+bx+c=0$ have three different roots *viz* α, β, γ.

Then since each of these values of x satisfies the equation.

$$a\alpha^2+b\alpha+c=0\ldots\ldots\mathbf{1}$$

$$a\beta^2+b\beta+c=0\ldots\ldots\mathbf{2}$$

$$a\gamma^2+b\gamma+c=0\ldots\ldots\mathbf{3}$$

Subtracting **2** from **1**

$a(\alpha^2-\beta^2)+b(\alpha-\beta)=0$ [Art. 21 A. Axiom.

or $a(\alpha+\beta)(\alpha-\beta)+b(\alpha-\beta)=0$ [Art 16. C

or $(\alpha-\beta)\{a(\alpha+\beta)+b\}=0$ [Art. 17.

Now, if the product of two factors be zero, one or both of the factors must be zero. [Axiom.

In the above $\alpha-\beta$ is not zero. $\because$ α and β are different.

$\therefore$ $a(\alpha+\beta)+b=0\ldots\ldots\ldots\ldots\mathbf{4}$

Similarly subtracting **3** from **1**, and reasoning that $\alpha-\gamma$ is not zero, we have $a(\alpha+\gamma)+b=0\ldots\ldots\mathbf{5}$

Subtracting **5** from **4**,

$$a(\beta-\gamma)=0 \ldots \ldots \ldots\ldots \mathbf{6}$$

But a is not zero. For if it were, the equation will become one of the first degree [$0.x^2=0$ $\therefore$ it vanishes.

Nor is $\beta-\gamma=0$, for if it were, β would be equal to γ, which we have assumed not to be the case.

$\therefore$ $a(\beta-\gamma)$ cannot be zero.

Therefore statement **6** is absurd.

Therefore the quadratic equation cannot have more than two roots.

25. Let α and β be the roots of the equation $ax^2+bx+c=0$.

$$\left.\begin{aligned} i.\,e.\ \alpha &= \frac{-b+\sqrt{b^2-4ac}}{2a} \\ \text{and}\quad \beta &= \frac{-b-\sqrt{b^2-4ac}}{2a} \end{aligned}\right\} \quad \text{[Art. 23.}$$

From these we have by adding

$$\alpha+\beta=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a} \quad \text{[Art. 21. B.}$$

$$=\frac{-2b}{2a}=-\frac{b}{a} \quad (A)$$ This is also established from Art. 24. (4).

Multiplying, we get

$$\alpha\beta=\frac{-b+\sqrt{b^2-4ac}}{2a}\times\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2} \quad \text{[Art. 12.}$$

$$=\frac{b^2-(b^2-4ac)}{4a^2} \quad \text{[Art. 9.}$$

$$=\frac{b^2-b^2+4ac}{4a^2}=\frac{4ac}{4a^2}$$ [Art. 3.

This is also neatly proved as follows :

From Art. 24. (1) and (2), by adding $a(\alpha^2+\beta^2)+b(\alpha+\beta)+2c=0.$

$$a\left(\overline{\alpha+\beta}^2-2\alpha\beta\right)+b(\alpha+\beta)+2c=0$$ [Art. 10. A and 21. A.

Substituting the value of 25. A.

$$a\left\{\left(-\frac{b}{a}\right)^2-2\alpha\beta\right\}+b\left(-\frac{b}{a}\right)+2c=0$$

$$a\,\frac{b^2}{a^2}-2a.\ \alpha\beta-\frac{b^2}{a}+2c=0$$

$$-2a.\ \alpha\beta=-2c.$$

$$\therefore\ \alpha\beta=\frac{c}{a}.$$

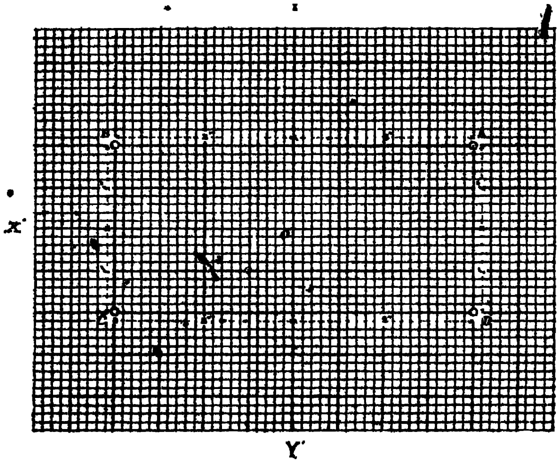
26. To find the equation where roots are given :—
Let α and b the given roots.

$$\therefore\ (x-\alpha)\ (x-b)=0$$

$\therefore\ x^2-(\alpha+\beta)x+\alpha b=0$ is the required equation.

27. Graphical representation of functions.

If two straight lines X O X' and Y O Y' are drawn in the plane of the paper, the position of any point in the plane can be known if the perpendicular distance of the point from these two straight lines be known. The straight lines are for convenience taken at right angles to one another.



Thus if the point be known to be 1″ from XOX′ and 2″ from YOY′, it must be situated at one of the positions A, B, C, D shown in the figure

To fix the position of the point uniquely, recourse is had to the following conventions.

1. Distances from Y O Y′ measured towards its right hand side along O X′ are reckoned positive.

2. Distances from Y O Y′ measured towards its left hand side along O X′ are reckoned negative.

3. Distances from X O X′, measured above it along OY are reckoned positive.

4 Distances from X O X′, measured below it along O Y are reckoned negative.

In accordance with these conventions, therefore, we have the following results :—

	Distance from YOY′.	Distance from XOX′.
For the point A	+ 2″	+ 1″
For the point B	− 2″	+ 1″

For the point C	$-2''$	$-1''$
For the point D	$+2''$	$-1''$

Using the scale $1''=1$ unit ;

These points are represented as $(2, 1)$, $(-2, 1)$, $(-2 -1)$ and $(2, -1)$ respectively.

Note :—The distance from Y O Y′ is written first, from XOX′ next.

If P be the point (x, y), x is called the *abscissa* and y the *ordinate* of the point P. Each of them is also called a *co-odinate* of P.

To plot the point P, measure off OM along OX (or OX′, if x is negative). From M draw MP at right angles to OX, making MP$=y$ units. [This should be drawn above XOX′ if y is positive, and below, if negative].

The relation between two quantities, one of which varies in some manner with the other, can, with advantage, be exhibited by means of figures drawn on squared or ruled papers. These figures are called *graphs* :

N. B.—A graph is really a line (or lines) straight or curved drawn specifically with reference to a given problem on small equal squares formed by the intersection of two sets of straight lines at exactly equal distances to two fixed straight lines at right angles—Ruled square papers are used to save the construction of the small squares.

Take, for example, the relation $y=x$.

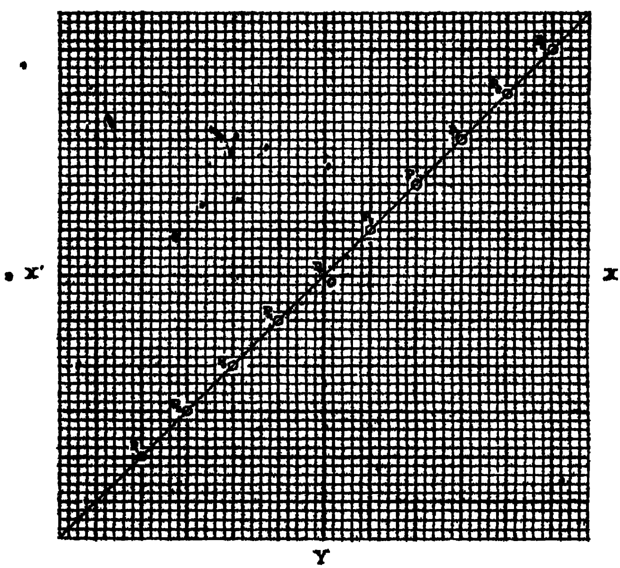
Tabulating the values we have

When $x=$	-4	-3	-2	-1	0	1	2	3	4	5
$y=$	-4	-3	-2	-1	0	1	2	3	4	5

Marking the points

$P_1(-4, -4)$, $P_2(-3, -3)$, $P_3(-2, -2)$, $P_4(-1, -1)$, $P_5(0,0)$,

$P_6(1,1)$, $P_7(2,2)$, $P_8(3,3)$, $P_9(4,4)$ and $P_{10}(5,5)$, and joining them we get the graph required.



Note. Graphs obtained from the equations of the first degree are straight lines.

Hence to determine the graph of such equation it is sufficient to join two points whose co-ordinates satisfy the given equation.

Note. (ii) Conversely, straight-line-graphs represent equations of the first degree, expressing the relation between two variables x and y. Such equations generally contain *both* x and y and are therefore indeterminate. In specific cases only *one* quantity appears but even then the other is not seen because its co-efficient is *zero*. Vide Exercise VII, Example 7. (i) and (ii).

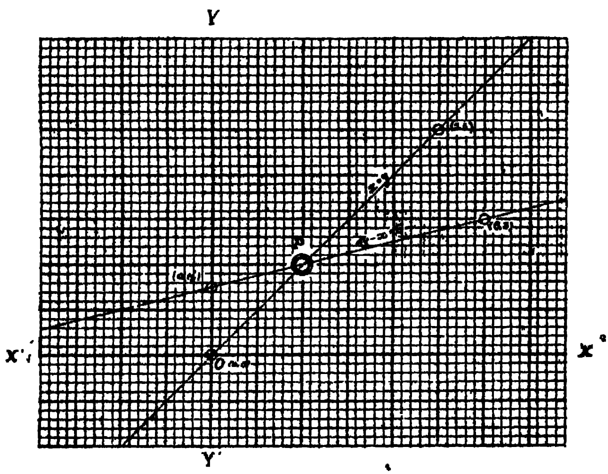
28. To Solve graphically a pair of simultaneous equations, draw the graph of each of them. The coordinates of the point of intersection give the values of x and y required.

Thus :—

Solve graphically $\left.\begin{array}{r} x=y. \\ 4y-x=6 \end{array}\right\}$

For the first equation $\left.\begin{array}{l} x=0 \\ y=0 \end{array}\right\}$ and $\left.\begin{array}{l} x=5 \\ y=5 \end{array}\right\}$ give two points on $x=y$. Joining them we get the graph

For the second $\left.\begin{array}{l} x=0 \\ y=1\frac{1}{2} \end{array}\right\}$ and $\left.\begin{array}{l} x=6 \\ y=3 \end{array}\right\}$ give two points on $4y-x=6$. Joining them the graph is obtained.

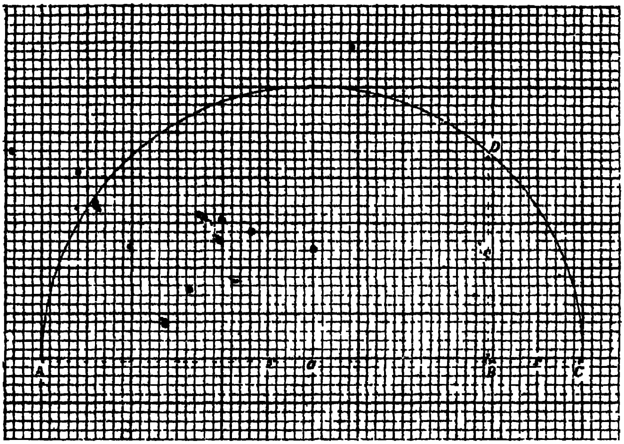


These st. lines intersect at the point P. From the figure the coordinates of P are (2,2). Therefore the solution of the simultaneous equation is

$$\left.\begin{array}{l} x=2 \\ y=2 \end{array}\right\}$$

29. To find the square root of any number graphically.

Let it be required to find the square root of 5.



Mark off on the squared paper AB = the given length, namely 5 (units), each unit being = ten sides of the small square. Produce it to BC, making BC = 1 (unit). Bisect AC at O. With centre O, radius OA describe a semicircle on AC. From B draw BD perpendicular to AC to meet the semicircle at D.

Then the length of BD gives the square root of 5. For, from geometry, $BD^2 = AB \times BC = 5 \times 1$ sq units

$$\therefore BD = \sqrt{5}.$$

From the figure we can easily read this length as 2·24,

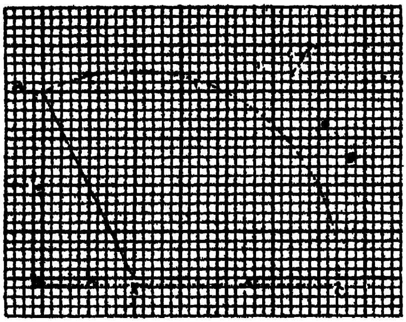
$$\therefore \sqrt{5} = 2{\cdot}24 \text{ approximately.}$$

There is one other method of obtaining the square root of numbers, based on Pythagoras Theorem, which states that the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares on its other two sides. This more special method is applicable to the cases of such numbers as can be written, as the sum of the squares of two other

numbers : For instance $5(=1^2+2^2)$, $10(=1^2+3^2)$, $13(=2^2+3^2)$ and so on.

The method is illustrated in the following figure for 5.

We can write 5 as $=1^2+2^2$.



On the squared paper mark off OA = 1. From O, draw OB at right angles to OA and = 2. Join AB. Then AB = $\sqrt{1^2+2^2}=\sqrt{5}$. To measure it, cut off AC from OA produced = AB, with a compass. Then AC can be directly read as 2·24.

$\therefore\ \sqrt{5}=2{\cdot}24$ approximately.

Exercise I.

1. Simplify the expression

$$5x-\{3x-(2x+3)\},$$

and find its value, when $x=-\frac{3}{4}$

2. Multiply $x+y+z$ by $x+y-z$.

3. Divide $a^4+b^4+2a^2b^2-2c^2d^2-c^4-d^4$

by $a^2+b^2-c^2-d^2$.

4. Solve $\frac{x}{3}+\frac{x}{7}=x-11$.

5 Express algebraically :—

(*a*) The number 5023 when $a=5$, $b=2$ and $c=3$.

(*b*) The excess of twice the difference between a and b over the product of c factors, each factor being the sum of nine times d and e.

6. Plot the following points :

(0,0) (5,1), (13, − 20), (− 5, − 10) and (− 2,0)

7. If $f(x)=x^2+2x+3$, find the values of

$f(0)$, $f(-1)$, $f(2)$, $f(4)$.

1 $5x-\{3x-(2x+3)\}$

$=5x-\{3x-2x-3\}$ [Art. 3.

$=5x-3x+2x+3$ [Art. 3.

$=4x+3$; $=4\times\left(-\frac{3}{4}\right)+3$ [Hyp.

$=-\frac{4\times 3}{4}+3$ [Art. 9.

$-3+3=0$.

2. Include $x+y$ in both expressions in a bracket,

thus, $\{(x+y)+z\}\{(x+y)-z\}$

$=(x+y)^2-z^2$ [Art. 12

$=x^2+y^2+2xy-z^2$ [Art. 10.

3. The dividend $=a^4+b^4+2a^2b^2-(c^4+d^4+2c^2d^2)$.

$=(a^2+b^2)^2-(c^2+d^2)^2$ [Art. 16 A.

$=(a^2+b^2+\overline{c^2+d^2})(a^2+b^2-\overline{c^2+d^2})$ [Art. 16. C

$=(a^2+b^2+c^2+d^2)(a^2+b^2-c^2-d^2)$ [Art. 3.

$\therefore$ Ans. $=a^2+b^2+c^2+a^2$.

4. $\dfrac{x}{3}+\dfrac{x}{7}=x-11$

Multiplying throughout by 21.

$7x+3x=21x-21\times 11$ [Art. 21 C.

$\therefore\ 7x+3x-21x=-21\times 11$ [Art. 21. A

or, $-11x=-21\times 11$

or, $-x=-21$ [Art. 21. D

or, $x=21$.

5. (a) $5023=5\times 1000+2\times 10+3$

$=1000a+10b+c$ [Art. 1.

(b) Twice the difference between a and b

$=2(a-b)$..... .... **(1)**

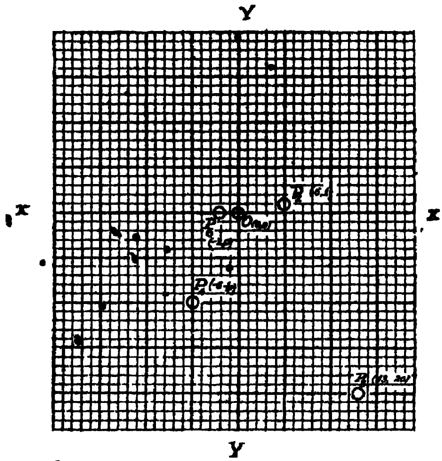
Sum of nine times d and e (and not nine times the sum of d and e) $=9d+e$, [and not $9(d+e)$].

The product of c such factors $=(9d+e)^c$ .....**(2)**

The excess of **(1)** over **(2)**.

$=2(a-b)-(9d+e)^c$.

6. [Art 27.



When $x =$	0	-1	2	4	
$x^2 =$	0	$(-1)^2 = 1$	$2^2 = 4$	$4^2 = 16$	Art. 9
$2x =$	0	$2x(-1) = -2$	$2 \times 2 = 4$	$2 \times 4 = 8$	Art. 9.
$3 =$	3	3	3	3	
$\therefore f(x) = x^2 + 2x + 3 =$	3	2	11	27	

[Art. 19.

Exercise II.

1. Simplify the expression

$$2\left[x - 3\left\{x - 4\left(x - \overline{\frac{5}{6}x - a}\right)\right\}\right]$$

2. Multiply $a^2 + ab + b^2$ by $a^2 - ab + b^2$.

3. (*a*) Divide $x^4 - x^3y - xy^3 + y^4$ by $x^2 + xy + y^2$, also $a^4 - 81$ by $a + 3$.

(*b*) Find the product of

$$x^{2r} - x^r y^p + \frac{y^{2p}}{4} \text{ by } x^{2r} + x^r y^p + \frac{y^{2p}}{4}.$$

4. Solve $\frac{bx}{a} - \frac{d}{c} = \frac{a}{b} - \frac{cx}{d}$.

5. A has Rs 160 and B has Rs 64, and each loses a certain sum. Then A has five times as much as B. What is the sum lost by each ?

6. For what value of a is
$x^3 - (a+6)x^2 + (6a+c)x + d$ divisible by $x - a$ without a remainder ?

7. Draw the graphs of the following equations
(i) $x = 10$, (ii) $y = -12$ and (iii) $x + 2y = 0$.

1. **Ans.** $= 2\left[x - 3\left\{x - 4\left(x - \frac{5}{6}x + a\right)\right\}\right]$ [Art. 3.

$= 2\left[x - 3x + 12x - 12 \times \frac{5}{6}x + 12a\right]$ [Art. 3.

$= 2[x - 3x + 12x - 10x + 12a]$.

$= 2 \times 12a = 24a$.

2. **Ans.** $= \{(a^2 + b^2) + ab\}\{(a^2 + b^2) - ab\}$ [Art. 3. Note Conv.

$= (a^2 + b^2)^2 - (ab)^2$ [Art. 12.

$= a^4 + b^4 + 2a^2b^2 - a^2b^2$ [Art. 10. A.

$= a^4 + b^4 + a^2b^2$.

3. (*a*) Dividend $= x^3(x-y) - y^3(x-y)$ [Art. 17.

$= (x^3 - y^3)\ (x-y).$ [Art. 17.

$= (x-y)\ (x^2 + xy + y^2)\ (x-y)$ [Art. 18. B.

$\therefore$ Ans. $= (x-y)\ (x-y).$

Again, $a^4 = 81 = (a^2)^2 - (9)^2$ [5 Conv.

$= (a^2+9)\ (a^2-9)$ [16. C.

$= (a^2+9)\ (a+3)\ (a-3).$ [16. C.

$\therefore$ Ans. $= (a^2+9)\ (a-3).$

(*b*) **Ans.** $= \left(x^{2r} + \frac{y^{2s}}{4}\right)^2 - (x^r y^s)^2$ [Art. 12.

$= (x^{2r})^2 + \left(\frac{y^{2s}}{4}\right)^2 + 2.\ x^{2r}.\frac{y^{2s}}{4} - (x^r y^s)^2$ [10. A.

$= x^{4r} + \frac{y^{4s}}{16} + \frac{x^{2r}y^{2s}}{2} - x^{2r}y^{2s}$ [Art. 5.

$= x^{4r} + \frac{y^{4s}}{16} - \frac{x^{2r}y^{2s}}{2}.$

4. $\frac{bx}{a} + \frac{cx}{d} = \frac{a}{b} + \frac{d}{c}$; $\frac{(bd+ac)x}{ad} = \frac{ac+bd}{bc}$

$\therefore\ \frac{x}{ad} = \frac{1}{bc}$ [21. D.

$\therefore\ x = \frac{ad}{bc}$ [Art. 21. C.

5. Let x = Sum lost by each in rupees.

A has $160 - x$, and B has $64 - x$.

$\therefore$ From the question $160 - x = 5\ (64 - x)$

$5x - x = 320 - 160$ [By transposition.

$4x = 160$ Art 21 A & B.

$x = 40.$

6. The remainder on division

$= a^3 - (a+6)a^2 + (6a+c)a + d$ [Art. 20.

$= a^3 - a^3 - 6a^2 + 6a^2 + ac + d$ [Art. 3.

$= ac + d.$

This must be zero *i. e.* $ac + d = 0$;

$$\therefore \quad a = -\frac{d}{c}$$

7 (i) $x = 10$.

This means that whatever be the value of y, x is always 10. Hence the table

$x =$	10	10	10	10
$y =$ (say)	-5	0	5	10

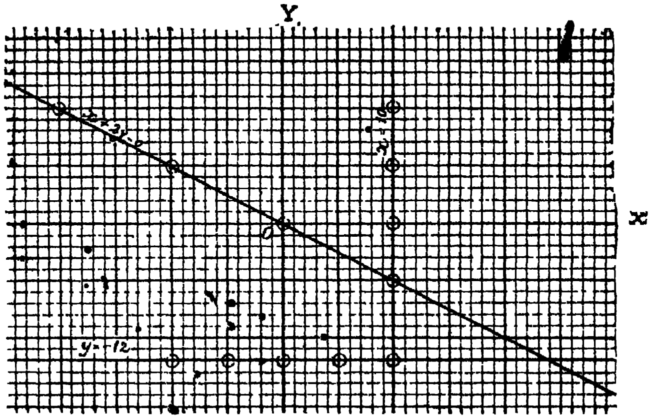
Plotting these and joining, we get the graph of $x = 10$.

(ii) Similarly, for $y = -12$ we have

$x =$	-10	-5	0	5	10
$y =$	-12	-12	-12	-12	-12

(iii) and for $x + 2y = 0$, or $x = -2y$ we have

$x =$	10	0	-10	-20
$y =$	-5	0	5	10



Exercise III.

1. Find the continued product of $(x-a)(x-b)(x-c)(x-d)$, hence deduce the value of $(x-2)^4$.

2. Divide $x^{\frac{3n}{2}} + x^{-\frac{3n}{2}}$ by $x^{\frac{n}{2}} + x^{-\frac{n}{2}}$.

3. Resolve into elementary factors $x^3 - 27$, $a^2 + b^2 - c^2 - 2ab$ and $a^3 - b^3$.

4. Show that $(a-b)(a+b-c)+(b-c)(b+c-a)+(c-a)(c+a-b)=0$.

5. Solve $\dfrac{x-1}{7}+\dfrac{23-x}{5}=7-\dfrac{4+x}{4}$.

6. A boy is one third the age of his father, and has a brother one sixth his own age; the ages of all three amount to 50 years. Find the age of each.

7. If $f(x)=ax^2+bx+c$ and $\phi(x)=a+bx+cx^2$, find the value of (1) $f(0)-\phi(0)$ and (2) $f(3)-\phi(2)$

8. Draw the graph of $y=x^2$.

1. In Art. 15, for a, b, c and d put $-a, -b, -c, -d$, respectively, and write them in the form $+(-a), +(-b), +(-c)$ and $+(-d)$ respectively. The answer is then

$$= x^4 - (a+b+c+d)x^3 + (ab+ac+ad+bc+bd+cd)x^2 - (abc+abd+acd+bcd)x + abcd.$$

Let $a=b=c=d=2$, then

$$(x-a)(x-b)(x-c)(x-d) = (x-2)(x-2)(x-2)(x-2)$$
$$= (x-2)^4.$$

$$\therefore \quad (x-2)^4 = x^4 - (2+2+2+2)x^3 + (2.2+2.2+2.2+2.2+2.2+2.2)x^2 - (2.2.2+2.2.2+2.2.2+2.2.2)x + 2.2.2.2.$$
$$= x^4 - 8x^3 + 24x^2 - 32x + 16.$$

2. $x^{\frac{3n}{2}} = (x^{\frac{n}{2}})^3$; $x^{-\frac{3n}{2}} = (x^{-\frac{n}{2}})^3$ [Art. 5. Conv.

$$\therefore \quad x^{\frac{3n}{2}} + x^{-\frac{3n}{2}} = (x^{\frac{n}{2}})^3 + (x^{-\frac{n}{2}})^3$$

$$= (x^{\frac{n}{2}} + x^{-\frac{n}{2}}) \left\{ (x^{\frac{n}{2}})^2 - x^{\frac{n}{2}} x^{-\frac{n}{2}} + (x^{-\frac{n}{2}})^2 \right\}$$ [Art. 18. A.

But $(x^{\frac{n}{2}})^2 = x^n$; and $x^{\frac{n}{2}} \cdot x^{-\frac{n}{2}} = x^{\frac{n}{2}-\frac{n}{2}} = x^0 = 1$

[Art. 5 and 8, note and 4.

and $(x^{-\frac{n}{2}})^2 = x^{-n}$ $\quad \therefore$ Ans. $= x^n - 1 + x^{-n}$.

3. (1) $x^3 - 27 = x^3 - 3^3 = (x-3)\{x^2+3x+3^2\}$ [18. B.

$$=(x-3)(x^2+3x+9)$$

(2) $a^2+b^2-2ab-c^2 = (a-b)^2 - c^2$ [16. B.

$$=(a-b+c)(a-b-c)$$

(3) $a^8 - b^8 = (a^4)^2 - (b^4)^2$. [Art. 5.

$= (a^4 + b^4)(a^4 - b^4)$ [16. C.

But $(a^4 - b^4) = (a^2)^2 - (b^2)^2 = (a^2 + b^2)(a^2 - b^2)$ [Art. 5

But $a^2 - b^2 = (a + b)(a - b)$ and 16. C.

$\therefore$ Ans $= (a^4 + b^4)(a^2 + b^2)(a + b)(a - b)$.

4. Since $a(x + y) = ax + ay$

$\therefore (a - b)(a + b - c) = (a - b)(a + b) - c(a - b)$

$= a^2 - b^2 - ac + bc$ [Art 17.

$\therefore$ Ans $= a^2 - b^2 - ac + bc + b^2 - c^2 - ab + ac + c^2 - a^2 - bc + ab$

$= 0$. [Art 12.

5. $\frac{x}{7} - \frac{1}{7} + \frac{23}{5} - \frac{x}{5} = 7 - \frac{4}{4} - \frac{x}{4}$

$\therefore \frac{x}{7} - \frac{x}{5} + \frac{x}{4} = 7 - 1 + \frac{1}{7} - 4\frac{3}{5} = 1\frac{2}{5} + \frac{1}{7}$

$\therefore \frac{20x - 28x + 35x}{7 \times 5 \times 4} = \frac{49 + 5}{5 \times 7}$

$\therefore \frac{27x}{4} = \frac{54}{1}$ or $\frac{x}{4} = 2$ [21. D.

$\therefore x = 8$.

6. Let x = age of the boy

$\therefore$ $3x$ = " " " father

$\therefore$ $\frac{x}{6}$ = " " " brother

$x + 3x + \frac{x}{6} = 50$ [Hyp.

$\therefore$ $\frac{25x}{6} = 50$ or $\frac{x}{6} = 2$. [21. D.

$\therefore$ $x = 12$ the age of boy [21. C.

$\therefore$ $3x = 36$ the age of the father

$\frac{x}{6} = 2$ " " " " brother.

7. (1) If $f(x)=ax^2+bx+c$

$f(0)=a\times 0^2+b\times 0+c$ [Art. 19.

$=c.$

$\phi(x)=a+bx+cx^2$

$\therefore \phi(0)=a+b\times 0+c.\ 0^2=a$ [Art. 19.

$\therefore f(0)-\phi(0)=c-a.$

(2) $f(x)=ax^2+bx+c$

$\therefore f(3)=a\ 3^2+b.\ 3+c$ [Art. 19

$=9a+3b+c.$

$\phi(x)=a+bx+cx^2$

$\therefore \phi(2)=a+2b+c.\ 2^2$ Art. 19

$=a+2b+4c.$

$\therefore f(3)-\phi(2)$

$=(9a+3b+c)-(a+2b+4c)$

$=8a+b-3c$

8. $y=x^2$.

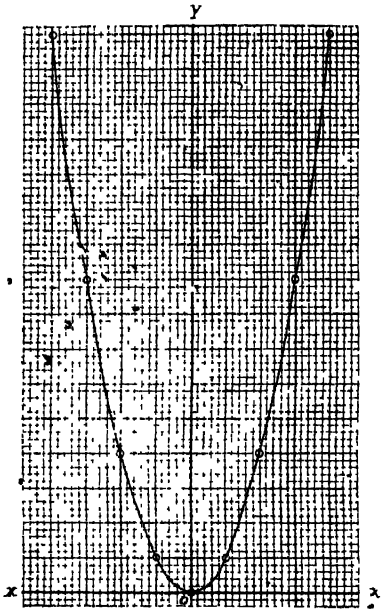
When $x=0$	± 1	± 2	± 3	± 4	± 5	...
$y=0$	1	4	9	16	25	.

Note. (i) The eqn. is not of the first degree and is not therefore a straight line.

(ii) There is no point below xox' for y cannot be negative. [There can be no real root of a negative quantity.]

Plotting the points and joining them by a smooth curve, the graph of $y=x^2$ is obtained.

The curve is known as a *parabola*.



Exercise IV.

1. (*a*) Show that the difference of the squares of any two consecutive numbers is equal to the sum of the numbers.

1. (*b*) Simplify $\sqrt[4]{\left(\frac{625 a^{24} b^{12} c^{4}}{81 x^{8} y^{16}}\right)}$

2. Show that $(a+b)^2-(c+d)^2+(a+c)^2-(b+d)^2$
$$= 2(a-d)\,(a+b+c+d).$$

3. Find the value of—
$(x+y+z)(x+y-z)(x+z-y)(y+z-x)$ when $z^2 = x^2+y^2$.

4. Solve (*a*) $\dfrac{2x-1}{x-2} = \dfrac{6x+4}{3x-4}$

(*b*) $(x+2)\,(x+4)\,(x+6) = x\,(x+5)\,(x+7)$.

5. If in a theatre $\frac{3}{8}$th of the seats are in the pit, $\frac{3}{10}$th in the lower gallery, $\frac{1}{5}$ in the upper. and there are 50 reserved seats : how many seats are there altogether ?

6. Thirty six copper coins consisting of pice and double pice are equivalent in value to a Rupee. How many of each kind are taken.

7. If $\phi(n) = 2n^2 - 5n + 3$ prove that
$$\phi(n+1) + \phi(n-1) - 2\phi(n) = 4.$$

8. A man starting at noon walks at the rate of 4 miles an hour. Draw a graph of his motion and read off as accurately as you can, the time when he is 26 miles from his starting point and the distance he has travelled in 2 hours 24 minutes.

1. **Def.** Consecutive numbers are such as differ from one another by unity. For example, 3 and 4 are consecutive numbers.

Let x and y be any two consecutive numbers.

Then, by def. $x - y = 1$.

$\therefore \quad (x-y)(x+y) = (x+y)$. [21. C. Axiom

or $x^2 - y^2 = x + y$ [Art. 12.

(*b*) Refer to Art 7.

$625 = 5^4$; and $81 = 3^4$.

$\sqrt[4]{a^{24}} = a^{\frac{24}{4}} = a^6$ [Art. 7.

$\sqrt[4]{b^{12}} = b^{\frac{12}{4}} = b^3$. [Art. 7.

Ans. $= \dfrac{5a^6b^3c}{3x^2y^4}$

2. $(a+b)^2 - (c+d)^2 = (a+b+c+d)(a+b-c-d)$ [16. C.

Similarly,

$(a+c)^2 - (b+d)^2 = (a+c+b+d)(a+c-b-d)$.

Adding we get

$(a+b+c+d)\{a+b-c-d+a+c-b-d\}$ [Art. 17.

$= 2(a-d)(a+b+c+d)$

3. $(x+y+z)(x+y-z)=(x+y)^2-z^2 \ldots 1$ [Art. 12.

$(x+z-y)(y+z-x)=(z+\overline{x-y})(z-\overline{x-y})$

$=z^2-(x-y)^2 \ldots 2$ [Art. 12.

But $z^2=x^2+y^2$ [Hyp.

∴ 1 $=x^2+y^2+2xy-(x^2+y^2)=2xy$ [Art. 10. A.

2 $=(x^2+y^2)-(x^2-2xy+y^2)=2xy$. [Art. 10. B.

∴ Ans. $=4x^2y^2$.

4. (a) Dividing the numerator by the denominator.

$$2+\frac{3}{x-2} = 2+\frac{12}{3x-4}$$

Subtracting 2 and dividing by 3, [21. A. D.

$$x-2 \quad 3x-4$$

∴ $3x-4=4x-8$ ∴ $4x-3x=8-4$. [21. D.

or $x=4$.

(b) The left hand member $=x^3+12x^2+44x+48$ [Art. 14. A.

The right hand member $=x(x^2+12x+35)$ [13. A.

∴ $x^3+12x^2+44x+48=x^3+12x^2+35x$ [Hyp.

∴ $44x-35x=-48$ [Art. 21. A.

∴ $9x=-48$

or $x=-\frac{48}{9}=-5\frac{1}{3}$.

5. Let x = number of seats required

∴ the number of seats in the pit $=\frac{3}{8}x$ [Hyp.

" " " " " " lower gallery $=\frac{3x}{10}$

" " " " " " upper " $=\frac{x}{5}$

number of reserved seats = 50.

∴ Total number of seats $=\frac{3}{8}x+\frac{3x}{10}+\frac{x}{5}+50$

From the question

$\therefore \quad \frac{3}{8}x + \frac{3x}{10} + \frac{x}{5} + 50 = x$

$\therefore \quad x - \frac{3}{8}x - \frac{3}{10}x - \frac{x}{5} = 50$ [21. A.

$\frac{40x - 15x - 12x - 8x}{40} = 50$

or $5x = 40 \times 50$ [21. C.

$\therefore \quad x = 40 \times 10 = 400.$

6. Let x = number of double pices

$\therefore \quad 36 - x$ = number of single pices.

$\therefore \quad 2x + 36 - x$ = value of one Rupee $= 64$

or $x + 36 = 64.$

$\therefore \quad x = 64 - 36 = 28$
$\therefore \quad 36 - x = 8$ [21. A.

7. $\phi(n) = 2n^2 - 5n + 3.$

$\therefore \quad \phi(n+1) = 2(n+1)^2 - 5(n+1) + 3.$ [Art. 19.

$= 2(n^2 + 2n + 1) - 5(n+1) + 3$ [Art. 10. A.

$= 2n^2 + 4n + 2 - 5n - 5 + 3$ [Art. 3.

$= 2n^2 - n \ldots\ldots\ldots\ldots$ **1**

Again, $\phi(n-1) = 2(n-1)^2 - 5(n-1) + 3$ [Art. 19.

$= 2(n^2 - 2n + 1) - 5(n-1) + 3$ [Art. 10. B

$= 2n^2 - 4n + 2 - 5n + 5 + 3$ [Art. 3.

$= 2n^2 - 9n + 10. \ldots\ldots\ldots$ **2**

$-2\phi(n) \quad = -4n^2 + 10n - 6 \ldots\ldots\ldots$ **3**

Adding **1**, **2** and **3**.

$\phi(n+1) + \phi(n-1) - 2\phi(n) = 4.$

8. Measure distance along OX, (one side of a square = 1 mile).

Measure times along OY, (10 sides of a square = 1 hour).

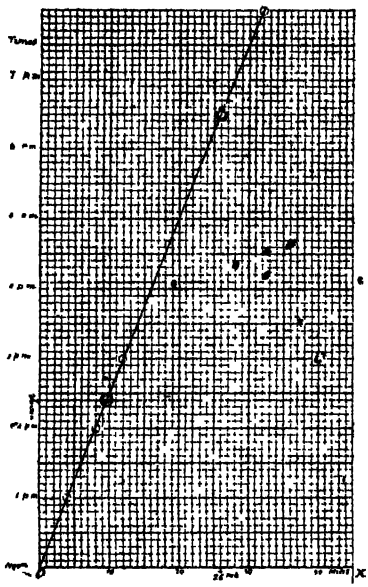
The following table gives the relation between the time and distance travelled.

Time	0 p.m.	1 p.m.	2 p.m.	3 p.m	8 p.m.
Distance	0	4 miles	8 mls	12 mls	32

Plotting the points (0,0) (4.1) (8.2) (12,3) *etc.*, due regard being paid to the different scales stated above and joining them the graph is obtained.

From the figure we see that corresponding to 26 miles time indicated is 6-30 p.m. and corresponding to 2 hours 24 minutes distance indicated is $9\frac{1}{2}$ miles.

These then give the required answers.



Exercise V.

1. Show that $7^{2n+1}+1$ is divisible by 8, when n is a positive integer.

2. Find the G.C.M. of

x^3-2x^2-2x-3 and x^3+2x^2+2x+1.

3. Resolve into elementary factors—

$x^2+9x+20$, $x^2-9x+20$, x^2+x-20, and x^2-x-20.

4. Solve the equations,

(i) $\dfrac{x-a}{b}+\dfrac{x-b}{a}=\dfrac{a^2+b^2}{ab}$.

(ii) $\begin{cases} 3x-2y=19 \\ 2x+3y=43. \end{cases}$

(iii) $x^2=x+20$.

5. A person bought a certain number of sheep for £94; having lost 7 of them, he sold $\frac{1}{4}$th of the remainder at prime cost of £20. How many sheep had he at first?

6. A and B began to play, the former with exactly $\frac{4}{5}$ of the sum which B had. After winning Rs 10, A had as much money as B. What had each at first?

7. Plot the points given by the table below and deduce the eqn of the graph which passes through them

$x=$	0	1	2	3	4
$y=$	·75	3·5	6·25	9	11·75

8. If a body falls freely under the acceleration g of gravity for t seconds, the space (in feet) it falls through is given by the formula $S=\frac{1}{2}gt^2$ where $g=32$.

Find the space a body under the acceleration of gravity falls through in 5 seconds.

Also find how long a body takes to fall through 144 feet.

1. $\because$ n is a positive integer, be it odd or even, $2n$ is always even.

$\therefore$ $2n+1$ is an odd number

Referring to Art. 20. corollary 1, wee see that a^m+b^m is divisible by $a+b$, when m is an odd number.

$\therefore$ $7^{2n+1}+1^{2n+1}$ is divisible by $7+1$ or 8.

N.B—1 raised to any power$=1$.

2. The first expression $= x^3 - 1 - 2x^2 - 2x - 2$

$= (x-1)(x^2+x+1) - 2(x^2+x+1)$ [18 B.

$= (x^2+x+1)(x-1-2)$ [Art 17.

$= (x^2+x+1)(x-3)$.

The second expression $= x^3 + 1 + 2x^2 + 2x$

$= (x+1)(x^2-x+1) + 2x(x+1)$ [18. A.

$= (x+1)(x^2-x+1+2x)$ [17.

$= (x+1)(x^2+x+1)$.

$\therefore$ G. C. M. $= x^2+x+1$.

3. 1st. $= x^2+4x+5x+20 = x(x+4)+5(x+4)$

$= (x+5)(x+4)$ [Art 17.

2nd $= x^2-4x-5x+20 = x(x-4)-5(x-4)$

$= (x-5)(x-4)$ [Art 18.

3rd $= x^2+5x-4x-20 = x(x+5)-4(x+5)$

$= (x+5)(x-4)$ [Art 18.

4th $= x^2-5x+4x-20 = x(x-5)+4(x-5)$

$= (x+4)(x-5)$ [Art 18.

Note—These examples illustrate the converse of Art 13.

4. (i) $\dfrac{ax - a + bx - b^2}{ab} = \dfrac{a^2+b^2}{ab}$.

$\therefore \quad ax - a^2 + bx - b^2 = a^2 + b^2$. [21. C.

$x(a+b) = 2(a^2+b^2)$ [21. B.

$\therefore \quad x = \dfrac{2(a^2+b^2)}{a+b}$. [21. D.

(*ii*) Multiply the 1st by 3, and the 2nd by 2.

$\therefore \quad 9x - 6y = 57 \ldots\ldots 1$

$4x + 6y = 86 \ldots\ldots 2$

Adding 1 and 2, $13x = 143$. $\therefore x = 11$. [21 D.

Substituting this value of x in the 2nd equation,

$3y = 43 - 22 = 21$.

$\therefore y = 7$.

(*iii*) $x^2 = x + 20$

This can be written as

$x^2 - x - 20 = 0$.

i.e., as $(x - 5)(x + 4) = 0$ [Ex V. 3.

$\therefore x = 5$ or -4 [Art. 23 B.

5. Let x = number of sheep he had at first.

Then $\frac{94}{x}$ = prime cost of each shep in £.

$x - 7$ = number of sheep after 7 were lost.

$\frac{x-7}{4}$ = number which he sold for £. 20.

But the price of each sheep × number sold = £20. [Hyp.

$\therefore \frac{94}{x} \times \frac{x-7}{4} = 20$

$\therefore 94x - 658 = 80x$. [Art. 21. C.

or $(94 - 80)x = 658$ [Art. 21 A.

or $14x = 658$

$\therefore x = 47$.

6. Let x = number of Rupees with which B began to play

$\therefore \frac{4}{9}x$ = " " " " " A. "

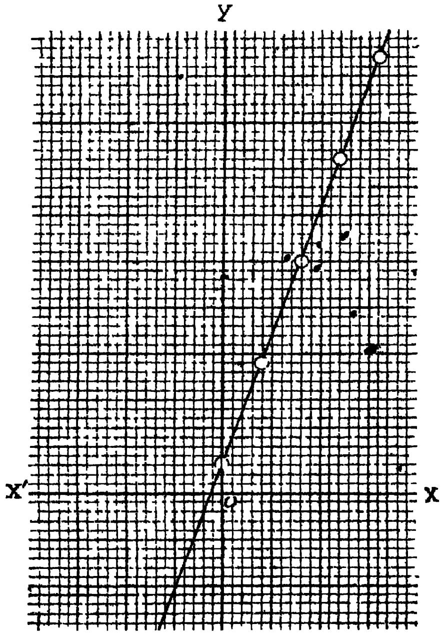
After the play A had $\frac{4}{9}x + 10$, and B, $x - 10$.

$\therefore \quad x - 10 = \frac{4}{9}x + 10$ [Hyp.

or $\frac{5}{9}x = 20$ [Art. 21. A.

or $x = \frac{9 \times 20}{5} = 36$ [Art. 21. C.

$\frac{4}{9}x = 16$.



The graph is, from the figure, a straight line.

$\therefore$ Its eqn. is of the first degree and is of the form

$$A\,x + B\,y + C = 0.$$

Since it passes through (0,·75) we have

$$\cdot 75\ B + C = 0 \ldots\ldots \mathbf{1}$$

Similarly because it passes through (1,3·5) and (3,9)

$$A + 3\cdot 5B + C = 0\ldots\ldots \mathbf{2}$$

$$3A + 9B + C = 0\ldots\ldots \mathbf{3}$$

From **1** $B = -\dfrac{C}{\cdot 75}$. [21. D.

Multiply **1** by 12 and subtract from **3**.

$$3A + 9B + C = 0$$

$$9B + 12C = 0$$

$$\therefore \quad 3A - 11C = 0$$

$$\therefore A = \frac{11C}{3}.$$

$\therefore$ The eqn becomes

$$\frac{11C}{3}x - \frac{4C}{3}y + C = 0.$$

$$\text{or } 11x - 4y + 3 = 0 \qquad [21.\ C.$$

$S = \frac{1}{2}gt^2 = 16t^2 = f(t)$.

Space in 5 seconds $= f(5) = 16 \times 5^2$ feet [19.

$= 400$ feet.

$$t^2 = \frac{s}{16}.$$

$$= \sqrt{\frac{s}{16}} = \psi(s).$$

Time for 144 feet $= \phi\ (144) = \sqrt{\frac{144}{16}} = \sqrt{9} = 3$ sec.

Exercise VI.

1. Simplify (a) $\dfrac{2^{n+4} - 2^n \times 2}{2^{n+2} \times 4}$

(b) $\sqrt{\{4 + \sqrt{(16x^2 + 8x^3 + x^4)}\}}$.

2. Reduce to their simplest forms.

(i) $(2ab^2 . 3d^{\frac{1}{n}})^3$.

(ii) $\dfrac{x^2 - 7x + 10}{x^2 - x - 2}$.

(iii) $\dfrac{x^4 - x^2 - 2x + 2}{2x^3 - x - 1}$.

3. Solve the equations,

(i) $x + \sqrt{2ax + x^2} = a$

(ii) $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{a}$

$\dfrac{1}{x} - \dfrac{1}{y} = \dfrac{1}{b}$

(iii) $25x^2 - 7x = 86$.

4. Resolve $4(xy + ab)^2 - (x^2 + y^2 - a^2 - b^2)^2$ into 4 factors.

5. If I buy oranges at 1s. 6d a dozen and $3\frac{1}{3}$ times as many apples at 4d a dozen, and after mixing them sell them at 1s a dozen and thereby gain 11s, how many dozens of each do I buy ?

6. A person sells a acres more than the m[th] part of his estate and there remain b acres less than the n[th] part. Of how many acres does the estate consist ?

7. If α and β be the roots of the equation

$$x^2 + px + q = 0$$

Find the equation whose roots are $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$

8. Draw the graph of $y = x^3$.

1. (a) $\dfrac{2^{n+4}-2^{n+1}}{2^{n+2}\times 2^2}=\dfrac{2^{n+1}\times 2^3-2^{n+1}}{2^{n+4}}$ [4 Conv.

$$=\frac{2^{n+1}(2^3-1)}{2^{n+1}\times 2^3}=\frac{2^3-1}{2^3}=\frac{7}{8}.$$

(b) $16x^2+8x^3+x^4=(4x)^2+2(4x)x^2+(x^2)^2$

$=(4x+x^2)^2.$ [Art. 16. A.

$\therefore$ Ans $=\sqrt{\{4+\sqrt{(4x+x^2)^2}\}}=\sqrt{\{4+4x+x^2\}}$

$=\sqrt{\{2^2+2\cdot 2\cdot x+x^2\}}=\sqrt{(2+x)^2}$ [Art. 16. A.

$=2+x.$

2. (i) Refer to Art 5.

Ans. $=8a^3b^6c^9d^{\frac{1}{2}}.$

(ii) Ans. $=\dfrac{(x-5)(x-2)}{(x-2)(x+1)}=\dfrac{x-5}{x+1}$ [Conv. 13 D &C.

(iii) Numerator $=x^2(x^2-1)-2(x-1).$

$=x^2(x+1)(x-1)-2(x-1)$ [16. C.

$=(x-1)\{x^2(x+1)-2\}$

$=(x-1)\{x^3+x^2-2\}.$

Denominator $=2x^3-2-x+1$

$=2(x^3-1)-1(x-1)$ [Art. 17.

$=2(x-1)(x^2+x+1)-1(x-1)$ [Art. 18. B.

$=(x-1)\{2(x^2+x+1)-1\}$ [Art. 17.

$=(x-1)(2x^2+2x+1)$

$\therefore$ Ans $=\dfrac{x^3+x^2-2}{2x^2+2x+1}.$

3. (*i*) $\sqrt{2ax+x^2}=a-x.$ [21. A.

Squaring $2ax+x^2=a^2-2ax+x^2.$ [21. E.

or $4ax=a^2.$

$\therefore\ x=\dfrac{a^2}{4a}=\dfrac{a}{4}.$

(*ii*) Adding the two equations,

$\dfrac{2}{x}=\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{a+b}{ab}$

$\therefore\ \dfrac{x}{2}=\dfrac{ab}{a+b}.$ [22. A.

$\therefore\ x=\dfrac{2ab}{a+b}.$ [21. C.

Subtracting the second from the first equation.

$\dfrac{2}{y}=\dfrac{1}{a}-\dfrac{1}{b}=\dfrac{b-a}{ab}.$ $\therefore$ by 22 A,

$\dfrac{y}{2}=\dfrac{ab}{b-a}$

$\therefore\ y=\dfrac{2ab}{b-a}=-\dfrac{2ab}{a-b}$ [$\because\ b-a=-a+b$

$=-(a-b)$

(*iii*) By Art 23, from the equation

$25x^2-7x-86=0$

$\therefore\ x=\dfrac{7\pm\sqrt{49+8600}}{50}=\dfrac{7\pm\sqrt{8649}}{50}$

$=\dfrac{7\pm93}{50}=\dfrac{100}{50}$ or $\dfrac{-86}{50}$

$=2$ or $-1\dfrac{18}{25}.$

4. $\{2(xy+ab)\}^2-(x^2+y^2-a^2-b^2)^2$

$=(2xy+2ab+x^2+y^2-a^2-b^2)\times(2xy+2ab-x^2$

$-y^2+a^2+b^2)$ [16. C.

Now the first factor $=(x^2+y^2+2xy)-(a^2+b^2-2ab)$

[Conv. 3.

$\quad ,, \;=(x+y)^2-(a-b)^2$ [Art. 16. A. & B.

$=(x+y+a-b)(x+y-a+b)$ [Art. 16. C.

2nd factor $=2ab+a^2+b^2-(x^2+y^2-2xy)$ [Conv. 3.

$=(a+b)^2-(x-y)^2$ [Art. 16. A. & B.

$=(a+b+x-y)(a+b-x+y)$ [Art. 16. C.

5. Let $x=$ No. of dozen of oranges bought.

$\therefore\ 3\frac{1}{2}x=$ no. of ,, ,, apples ,,

The oranges cost $1\frac{1}{2}x$ shillings.

,, apples cost $3\frac{1}{2}x\times\frac{1}{3}$ shillings.

$\therefore$ Total cost $=\frac{3x}{2}+\frac{7x}{6}$ shillings.

They are sold for $(3\frac{1}{2}x+x)\times 1$ or $\frac{9x}{2}$ shillings.

$\therefore$ by the question,

$$\frac{9x}{2}-\left(\frac{3x}{2}+\frac{7x}{6}\right)=11.$$

$\therefore\ \frac{11x}{6}=11\ \therefore\ x=6$ (oranges);

$3\frac{1}{2}x=21$ (apples).

6. Let x = number of acres required.

He sells $\frac{x}{m} + a$ and there remains $\frac{x}{n} - b$.

$\therefore$ by the question, $\frac{x}{m} + a + \frac{x}{n} - b = x$.

$$\text{or } x - \frac{x}{m} - \frac{x}{n} = a - b \qquad \text{[21. A.}$$

$$\text{or } \frac{(mn - m - n)x}{mn} = a - b.$$

$$\therefore \quad x = \frac{mn(a - b)}{mn - m - n}. \qquad \text{[21 C.}$$

7. By Art 25, $\alpha + \beta = -p$ and $\alpha\beta = q$.

Also the eqn. whose roots are $\frac{1}{\alpha}$, $\frac{1}{\beta}$ is

$$x^2 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)x + \frac{1}{\alpha\beta} = 0 \qquad \text{[Art. 26.}$$

$$\text{or} \quad x^2 - \left(\frac{\alpha + \beta}{\alpha\beta}\right)x + \frac{1}{\alpha\beta} = 0$$

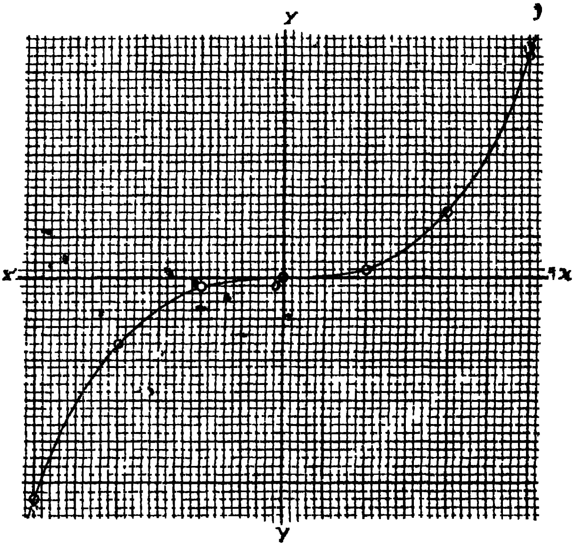
$$\text{or} \quad x^2 + \frac{p}{q}x + \frac{1}{q} = 0$$

$$\text{or} \quad qx^2 + px + 1 = 0.$$

8. Use for the y values one tenth of that for x.

when	$x = -3$	-2	-1	0	1	2	3	4
	$y = -27$	-8	-1	0	1	8	27	64

Plot these points and join them by a smooth curve



Exercise VII.

1. Show that the continued product of any four consecutive numbers together with unity is a square number.

2. (*a*) Show that $(x^2 - xy + y^2)^3 + (x^2 + xy + y^2)^3$ is divisible by $2x^2 + 2y^2$.

(*b*) Resolve into elementary factors

$$2xyz + x^2(y+z) + y^2(z+x) + z^2(x+y).$$

3. Simplify

(*a*) $$\frac{x^4 + 2x^2 + 9}{x^4 - 4x^3 + 4x^2 - 9}$$

(*b*) $$\frac{}{(a-b)(a-c)} + \frac{}{(b-a)(b-c)} + \frac{}{(c-a)(c-b)}$$

4. Solve

(i) $\dfrac{x+a}{x-a}=b$

(ii) $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=b.$

(iii) $5x+11y=146$
$11x+5y=110$

5. A crew which can pull at the rate of nine miles an hour finds that it takes twice as long to come up a river as to go down. At what rate does the river flow ?

6. Find two numbers in the proportion of 8·5 the product of which is 360.

7. If α and β be the roots of the equation

$ax^2+bx+c=0$, find the equation whose roots are

(i) $\dfrac{1+\alpha}{\beta}, \dfrac{1+\beta}{\alpha}$

(ii) α^2, β^2.

8. Find three cube roots of 1.

1. Let x, $x+1$, $x+2$, and $x+3$ be four consecutive numbers. [See Def. Exercise IV. 1 (a)

Their product $=x(x+1)(x+2)(x+3)$.

$=x(x+3)(x+1)(x+2)$ [$\because a\times b=b\times a$.

$=(x^2+3x)(x^2+3x+2)$ [13. A.

But $x^2+3x=(x^2+3x+1)-1$;

and $x^2+3x+2=(x^2+3x+1)+1$.

$\therefore$ Their product $=(x^2+3x+1)^2-1$ [Art. 12.

Add unity to their product,

$\therefore$ Result $=(x^2+3x+1)^2$ which is a square number.

2. (*a*) Refer to Art 20. Corollary 1.

Now, because in the given expression the index is 3, which is an odd number,

therefore it is divisible by their sum

$(x^2 - xy + y^2) + (x^2 + xy + y^2)$ which $= 2x^2 + 2y^2$.

(*b*) $2xyz + x^2z + y^2z + x^2y + xy^2 + z^2)(x+y)$

$= z\{2xy + x^2 + y^2\} + xy(x+y) + z^2(x+y)$ [Art. 17.

$= z(x+y)^2 + xy(x+y) + z^2(x+y)$ [Art. 16. A.

$= (x+y)\{z(x+y) + xy + z^2\}$ [Art. 17.

$= (x+y)\{z^2 + (x+y)z + xy\}$

$= (x+y)(z+x)(z+y)$ [Art. 13. A. Conv.

3. (*a*) Numerator $= x^4 + 6x^2 + 9 - 4x^2$

$= (x^2 + 3)^2 - (2x)^2$

$= (x^2 + 2x + 3)(x^2 - 2x + 3)$ [Art. 16. C.

Denominator $= (x^2)^2 - 2 .\ x^2 .\ 2x + (2x)^2 - 3^2$

$= (x^2 - 2x)^2 - 3^2$ [Art. 16. B.

$= (x^2 - 2x + 3)(x^2 - 2x - 3)$ [Art. 16. C.

Ans. $= \dfrac{x^2 + 2x + 3}{x^2 - 2x - 3}$.

(*b*) $b - a = -(a - b)$; $c - a = -(a - c)$

and $c - b = -(b - c)$ [Art. 3. Conv.

$(c - a)(c - b) = \{-(a - c)\} \times \{-(b - c)\}$

$= (a - c)(b - c)$ [Art. 9.

$\therefore$ The given expression $=$

$$\frac{1}{(a-b)(a-c)} - \frac{1}{(a-b)(b-c)} + \frac{1}{(a-c)(b-c)}$$

$$= \frac{(b-c) - (a-c) + (a-b)}{(a-b)(a-c)(b-c)} = 0.$$ [$\because$ the numerator $= 0$.

4. (*i*) If in the numerator and denominator the terms are the same, but the connecting signs different, apply [Art. 22. E.]

Thus $\dfrac{x+a+x-a}{x+a-(x-a)}=\dfrac{b+1}{b-1}$.

or $\dfrac{2x}{2a}=\dfrac{b+1}{b-1}$; *i.e.*, $\dfrac{x}{a}=\dfrac{b+1}{b-1}$ [2 being cancelled.

$\therefore\ x=\dfrac{a(b+1)}{b-1}$.

4. (*ii*) Here, also apply the Art. 22 E.

$\therefore\ \dfrac{2\sqrt{a+x}}{2\sqrt{a-x}}=\dfrac{b+1}{b-1}$; or $\dfrac{\sqrt{a+x}}{\sqrt{a-x}}=\dfrac{b+1}{b-1}$.

Squaring, $\dfrac{a+x}{a-x}=\dfrac{b^2+2b+1}{b^2-2b+1}$

Again apply the same article, viz. 22. E.

$\therefore\ \dfrac{a}{x}=\dfrac{b^2+1}{2b}$ or $\dfrac{x}{a}=\dfrac{2b}{b^2+1}$ [22 A.

$\therefore\ x=\dfrac{2ab}{b^2+1}$ [21. C.

(*iii*) Add the two equations.

$16\,(x+y)=256$.

$\therefore\ x+y=16$.

Multiplying by 5, $5x+5y=80$.. 1

Subtract 1 from the first equation

$\therefore\ 6y=66$: $\therefore\ y=11$.

But $x+y=16$ $\therefore\ x=5$.

5. Let the river flow x miles in one hour.

Now, in one hour the crew pulls nine miles.

$\therefore$ It takes one hour to go down $x+9$ miles, which to come up, it would take two hours by the question.

In two hours the crew pulls 18 miles. But the stream being against at the rate of x miles an hour, they really move over $18-2x$ miles.

$\therefore \quad x+9=18-2x$; or $3x=9$; or $x=3$ miles.

6. Let the two numbers be $8x$ and $5x$.

$\therefore \quad 8x \times 5x=360$; or $40x^2=360$

$\therefore \quad x^2=9$, or $x=\pm 3$. [21. E.

$\therefore \quad 8x=24$ or -24

and $5x=15$ or -15.

7. By art 25, $\alpha+\beta=-\frac{b}{a}$, $\alpha\beta=\frac{c}{a}$

(*i*) Also, by art 26 the required equation is

$$x^2-\left\{\frac{1+\alpha}{\beta}+\frac{1+\beta}{\alpha}\right\}x+\frac{(1+\alpha)}{\beta}\times\frac{1+\beta}{\alpha}=0.$$

or Multiplying by $\alpha\beta$

$$\alpha\beta x^2-\{\alpha+\alpha^2+\beta+\beta^2\}x+1+\alpha+\beta+\alpha\beta=0 \quad [13.\ A.$$

$$\text{or } \frac{c}{a}x^2-\left\{(\alpha+\beta)+(\alpha+\beta)^2-2\alpha\beta\right\}x+1-\frac{b}{a}+\frac{c}{a}=0$$

[$\because \alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$$\text{or } \frac{c}{a}x^2-\left\{-\frac{b}{a}+\frac{b^2}{a^2}-\frac{2c}{a}\right\}x+\frac{(a-b+c)}{a}.$$

or $acx^2+(ab-b^2+2ac)x+a^2-ab+ac=0$.

(ii) Similarly, the required equation

is $x^2-(\alpha^2+\beta^2)x+\alpha^2\beta^2=0$ [Art 26.

or $x^2-\{(\alpha+\beta)^2-2\alpha\beta\}x+(\alpha\beta)^2=0$

or $x^2-\left\{\frac{b^2}{a^2}-\frac{2c}{a}\right\}x+\frac{c^2}{a^2}=0$ [Art 25.

or $a^2x^2-(b^2-2ac)x+c^2=0$

8. Let $x^3 = 1$.

Then x is by definition the required root.

$\therefore \quad x^3 - 1 = 0$

$\therefore \quad (x-1)(x^2+x+1)=0$ [Art. 18. B.

$\therefore \quad x-1=0$ or $x^2+x+1=0$ [Art. 23. B.

$x-1=0$ gives $x=1$.

$x^2+x+1=0$ gives

By Art. 23. A, $x = \dfrac{-1 \pm \sqrt{1-4}}{2}$ $\left[\begin{array}{l} \text{Now } \sqrt{1-4} = \sqrt{-3} \\ = \sqrt{-1}\,\sqrt{3} \\ = i\sqrt{3} \end{array} \right.$

$$\frac{-1+i\sqrt{3}}{2} \text{ and } \frac{-1-i\sqrt{3}}{2}.$$

Where i stands for the imaginary value $\sqrt{-1}$.

Note. This $\sqrt{-1}$ is the typical *imaginary* quantity. It has no real numerical existence for no number whether positive or negative being squared can yield a negative value [Art. 9.]. Conversely there is no number which is the square root of -1.

It is clear therefore that generally the square root of any negative quantity is an imaginary quantity—not only of $\sqrt{-1}$ but of $\sqrt{-5}$ $\sqrt{-9}$ etc.

Exercise VIII.

1. If n be any positive integer show that $(ab)^n - (bc)^n + (cd)^n - (da)^n$ is divisible by $ab - bc + cd - da$

2. Resolve into elementary factors.

(i) $(1+a)^2(1+c^2) - (1+c)^2(1+a^2)$,

(ii) $x^4 - (a+b)x^3 + (a+b)abx - a^2b^2$.

(iii) $x^4 + x^2 + 1$.

3. Simplify $\dfrac{m^2 - mn + n^2}{m^3 - 3mn(m-n) - n^3} \times \dfrac{m^2 - n^2}{m^3 + n^3}$

4. Solve (i) $\sqrt{x-16} + \sqrt{x} = 8$.

(ii) $\dfrac{ax - b^2}{(ax)^{\frac{1}{2}} + b} + c = \dfrac{ax^{\frac{1}{2}} - b}{n}$

(iii) $3x + 2y = 13$
$3y + 2z = 8$
$3z + 2x = 9$

(iv) $(x-1)^2 = 16$.

5. A person has just a hours at his disposal. How far may he ride in a coach which travels b miles an hour so as to return home in time walking back at the rate of c miles an hour?

6. A man travelled certain distance at the rate of seven miles an hour. He then found that if he had not travelled so fast by two miles an hour, he should have been six hours longer in performing the same journey. Find the distance.

7. Draw the graph of : $3x - 4y = 12$.

8. If $\phi(n) = \dfrac{n(n+1)}{2}$, find the value of $\phi(10) - \phi(9)$.

1. The given expression $= a^n b^n - b^n c^n + c^n d^n - d^n a^n$

$$= b^n(a^n - c^n) - d^n(a^n - c^n)$$

$$= (b^n - d^n)(a^n - c^n) \qquad \text{[Art. 17.}$$

But $a^n - c^n$ is divisible by $a - c$, [Art. 20 Cor. 2.

also $b^n - d^n$ is divisible by $b - d$ [Art 20. Cor. 2.

$\therefore$ The whole expression is divisible by

$(a-c)(b-d)$ or $ab - bc + cd - da$.

2 (i) $=(1 + a^2 + 2a)(1 + c^2) - (1 + c^2 + 2c)(1 + a^2)$ [10 A.

$$= (1 + a^2)(1 + c^2) + 2a(1 + c^2) - (1 + c^2)(1 + a^2) - 2c(1 + a^2)$$

$$= 2a(1 + c^2) - 2c(1 + a^2)$$

$$= 2a - 2c + 2ac^2 - 2a^2c$$

$$= 2(a - c) - 2ac(a - c)$$

$$= 2(a - c)(1 - ac) \qquad \text{[Art. 17.}$$

(ii) $= x^4 - a^2b^2 - (a + b)x^3 + (a + b)abx$

$$= (x^2 + ab)(x^2 - ab) - x(a + b)(x^2 - ab) \qquad \text{[16. C. and 17.}$$

$$= (x^2 - ab)\{x^2 + ab - x(a + b)\}$$

$$= (x^2 - ab)(x - a)(x - b) \qquad \text{[Art. 13. D. Conv.}$$

(iii) $x^4 + 2x^2 + 1 - x^2 = (x^2 + 1)^2 - x^2$ [Art. 16. A.

$$= (x^2 + 1 + x)\{(x^2 + 1) - x\} \qquad \text{[Art. 16. C.}$$

or $(x^2 + x + 1)(x^2 - x + 1)$

Note 1.—The process of this example is the converse of solution 2, Ex. II.

3. Ans. $= \dfrac{m^2 - mn + n^2}{(m - n)^3} \times \dfrac{(m + n)(m - n)}{(m + n)(m^2 - mn + n^2)}$ [11. B. 16.C. and 18.A.

$$= \frac{1}{(m - n)^2}.$$

4. (*i*) $\sqrt{x-16}+\sqrt{x}=8 \ldots\ldots\ldots 1$ [Hypothesis

But $(x-16)-x=-16 \quad .2$ [Identity.

Divide **2** by **1**.

Note 2.—This is really a particular case of the general expression $x^{4n}+x^{2n}y^{2n}+y^{4n}$ whose factors are $x^{2n}+x^n y^n+y^{2n}$ and $x^{2n}+x^n y^n+y^{2n}$. Here $n=1$ and $y=1$.

$\therefore \sqrt{x-16}-\sqrt{x}=-2 \ldots\ldots 3$ [16. C.

Subtract **3** from **1** . $\therefore 2\sqrt{x}=10$, or $\sqrt{x}=5$

$\therefore x=25$ [21. E.

Note—*Identity* means *Sameness*. Whatever value may be assigned to x, $(x-16)-x=-16$. Hence the two sides of this equation are said to be identical.

(*ii*) $\because ax-b^2=\{(ax)^{\frac{1}{2}}+b\}\{(ax)^{\frac{1}{2}}-b\}$ [16 C.

$\therefore \quad (ax)^{\frac{1}{2}}-b+c=\dfrac{(ax)^{\frac{1}{2}}-b}{n}$ [From the equation

$\therefore \quad \{(ax)^{\frac{1}{2}}-b\}\left(1-\dfrac{1}{n}\right)=-c$ [By transposition and Art 18.

$\therefore (ax)^{\frac{1}{2}}-b=\dfrac{-cn}{n-1}$

$\therefore (ax)^{\frac{1}{2}}=b-\dfrac{cn}{n-1}$

$ax=\left\{b-\dfrac{cn}{n-1}\right\}^2. \quad \therefore x=\dfrac{1}{a}\left\{b-\dfrac{cn}{n-1}\right\}^2.$

(*iii*) Adding the equations,

$5(x+y+z)=30$

$\therefore x+y+z=6 \ldots\ldots\ldots 1$

$\therefore 3x+3y+3z=18$ [Multiplying by 3.

But $y+3z=5 \ldots\ldots\ldots .2$

Multiplying 2 by 3, $3y+9z=15$.

But $3y+2z=8$

$\therefore 7z=7$ or $z=1$

Substitute the value of z in 2 $\therefore y=5-3=2$.

Substitute in the third equation the value of z.

$\therefore 2x=9-3=6$ or $x=3$.

(iv) Extracting the sq. root.

$x-1=\pm 4$ [Art. 21. E.

$\left.\begin{array}{l} x-1=4 \\ x=5 \end{array}\right\}$ $\begin{array}{l} \text{or } x-1=-4 \\ \text{or } x=-3. \end{array}$

5. Let x = number of miles the person may ride.

$\therefore$ it would take him $\frac{x}{b}$ hours to ride in a coach, and $\frac{x}{c}$ hours to walk back.

$\therefore$ By the question, $\frac{x}{b}+\frac{x}{c}=a$ $\therefore x=\frac{abc}{b+c}$.

6. Let x = distance required in miles.

$\therefore$ the time taken to perform the journey $=\frac{x}{7}$ hours.

If he had reduced his speed by two miles, he would have taken $\frac{x}{5}$ hours to perform the same journey.

$$\therefore \frac{x}{5}-\frac{x}{7}=6$$

$$\text{or } \frac{2x}{35}=6$$

$$\therefore x=35\times 3=105.$$

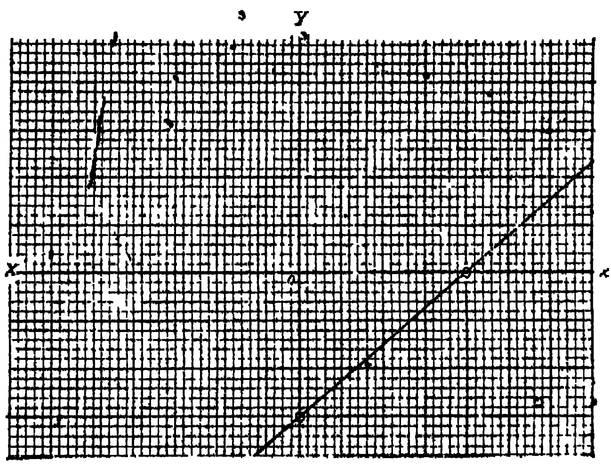
7. (*i*) The equation is of the first degree.

$\therefore$ it represents a straight line.

If $x=0$, $y=-3$.

If $x=4$, $y=0$

Plotting these points and joining them we get the graph required.



8. $\phi(10)=\dfrac{10\times(10+1)}{2}=5\times 11=55.$

$\phi(9)=\dfrac{9\times(9+1)}{2}=5\times 9=45.$ [Art. 19.

$\therefore$ $\phi(10)-\phi(9)=10.$

Exercise IX.

1. If any three consecutive whole numbers be taken, prove that the sum of the squares of the greatest and least is greater by 2 than twice the square of the middle number.

2. Show that $(x+y+z)^3-(x^3+y^3+z^3)$

$$=3(x+y)(x+z)(y+z).$$

3. (*a*) Resolve x^4+4 into factors.

(*b*) Find the G. C. M. of $x^3+ax^2-axy-y^3$ and $x^4+2x^3y-a^2x^2+x^2y^2-2axy^2-y^4$.

4. Solve (*i*) $\sqrt{x-5}+\sqrt{x+7}=6$

(*ii*) $\left.\begin{array}{l} my+nx=axy \\ ny+mx=bxy \end{array}\right\}$

(*iii*) $x+35=70x^2$.

5. A certain fraction becomes $\frac{1}{3}$ if 1 be added to its numerator; and if 1 be added to its denominator, it becomes $\frac{1}{4}$: what is the fraction?

6. The length of a certain rectangular is to its breadth as 6 : 5. One-sixth part of the area being planted, there remains for ploughing 625 square yards. Show that the dimensions of the field are 30 and 25.

7. If α and β be the roots of the equation

$$ax^2+bx+c=0$$

Find the equation whose roots are α^3 and β^3.

8. A and B, travelling at 8 and 12 miles an hour respectively, bicycle towards one another from two places 50 miles apart, starting at the same time. Find graphically when and where they meet, and when they are 10 miles from one another.

1. Let $x-1$, x, $x+1$ be 3 consecutive numbers, of which $x-1$ is the least and $x+1$ the greatest. [See Def. Ex. IV. I.

Now $(x-1)^2+(x+1)^2=(x^2-2x+1)+(x^2+2x+1)$ [10.B&A

$=2(x^2+1)$ *i.e.* $(x+1)^2+(x-1)^2$ is greater than $2x^2$ by 2.

2. $(x+y+z)=(x+y)+z$.

$\therefore$ $(x+y+z)^3=(x+y)^3+z^3+3z(x+y)\{(x+y)+z\}$ [11.A.

$=x^3+y^3+3xy(x+y)+z^3+3z(x+y)\{x+y+z\}$

Subtract $x^3+y^3+z^3$, therefore the remainder

$=3xy(x+y)+3z(x+y)(x+y+z)$

$=3(x+y)\{xy+z(x+y+z)\}$ [Art. 17.

$=3(x+y)\{xy+z(x+y)+z^2\}$

$=3(x+y)(x+z)(y+z)$ [Art. 13. A. Conv.

3. (*a*) $x^4+4=x^4+4x^2+4-4x^2$

$=(x^2+2)^2-(2x)^2$. [16. A.

$=(x^2+2x+2)(x^2-2x+2)$ [16. C

Note.—This is a particular case of the factorization of the general form x^4+4a^4 which is resolved into $(x^2+2ax+2a)$ $(x^2-2ax+2a)$ a being $=1$. Similarly by giving to a in succession the integral values 2, 3, 4 *etc* we find the factor of x^4+64, x^4+324, x^4+1024 &c.

(*b*) 1*st* quantity $=x^3-y^3+ax^2-axy$.

$=(x-y)(x^2+xy+y^2)+ax(x-y)$ [18. B.

$=(x-y)\{x^2+xy+y^2+ax\}$ [Art. 17.

2*nd* quantity $=x^4+2x^3y+x^2y^2-(a^2x^2+2axy^2+y^4)$.

[Art. 3. Conv.

But $x^4+2x^3y+x^2y^2=(x^2+xy)^2$ [16. A.

and $a^2x^2+2axy^2+y^4=(ax+y^2)^2$ [16. A.

$\therefore$ 2*nd* quantity $=(x^2+xy)^2-(ax+y^2)^2$

$=(x^2+xy+ax+y^2)(x^2+xy-ax-y^2)$ [16. C.

$\therefore$ Ans. $=x^2+xy+ax+y^2$.

4. (*i*) $\sqrt{x-5}+\sqrt{x+7}=6$..............................1

$(x-5)-(x+7)=-12$..............................2

[Identity.

$\therefore$ Dividing **2** by **1**, $\sqrt{x-5}-\sqrt{x+7}=-2$...**3**

Adding **1** and **3**, $2\sqrt{x-5}=4$.

or $\sqrt{x-5}=2$.

$\therefore$ $x-5=4$; or, $x=9$ [Art. 21. E.

(*ii*) Dividing the equations by xy,

$$\frac{my}{xy}+\frac{nx}{xy}=a\text{; or }\frac{m}{x}+\frac{n}{y}=a\ldots\mathbf{1}$$

Similarly $\dfrac{n}{x}+\dfrac{m}{y}=b$............**2**

Multiply **1** by m, and **2** by n;

$$\therefore\quad \frac{m^2}{x}+\frac{mn}{y}=am.\text{ and }\frac{n^2}{x}+\frac{mn}{y}=nb.$$

Subtracting the latter from the former,

$$\frac{m^2-n^2}{x}=ma-nb.\quad\therefore\text{ by 22. A}$$

$$\frac{x}{m^2-n^2}=\frac{1}{ma-nb}\quad\therefore\quad x=\frac{m^2-n^2}{ma-nb}.$$

Similarly multiplying **1** by n, and **2** by m we get

$$\therefore\ y=\frac{m^2-n^2}{mb-na}.$$

4. (*iii*) $70x^2-x-35=0$

$$\therefore\ x=\frac{1\pm\sqrt{1+35\times70\times4}}{2\times70}.$$ [23. A.

$$=\frac{1\pm\sqrt{9801}}{140}=\frac{1\pm99}{140}=\frac{100}{140}\text{ or }-\frac{98}{140}$$

$$=\frac{5}{7}\text{ or }-\frac{7}{10}.$$

5. Let the numerator of the required fraction $= x$
and „ denominator „ $= y$.

$$\frac{x+1}{y} = \frac{1}{3} \quad \ldots\ldots\ldots 1$$

$$\frac{x}{y+1} = \frac{1}{4} \quad \ldots\ldots 2 \qquad \text{[Hyp.}$$

$$\therefore \quad \left.\begin{array}{l} 3x - y = -3 \\ 4x - y = 1 \end{array}\right\} \qquad \therefore \quad \begin{array}{l} x = 4. \\ y = 15. \end{array}$$

or the required fraction $= \frac{4}{15}$.

6. The length of the field in yds $= 6x$, suppose.

$5x =$ breadth of the field.

area $= 30x^2$.

One-sixth being planted there remains $1 - \frac{1}{6}$ or, $\frac{5}{6}$ for ploughing

$$\therefore \quad \frac{5}{6} \times 30x^2 = 625 \qquad \text{[Hyp.}$$

$$25x^2 = 625$$

$$x^2 = 25 \qquad \text{[21. C.}$$

or $x = 5$.

Whence $6x = 30$, $5x = 25$.

By art. 25

$$\alpha + \beta = -\frac{b}{a}$$

$$\alpha\beta = \frac{c}{a};$$

and, by art 26, the required equation is

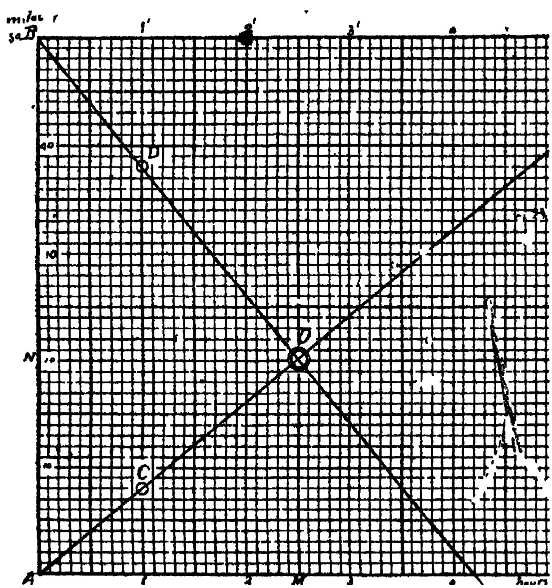
$$x^2 - (\alpha^3 + \beta^3)x + \alpha^3\beta^3 = 0$$

or $\quad x^2 - \{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)\}x + \alpha^3\beta^3 = 0 \qquad$ [Art. 11. A.

or $\quad x^2 - \left\{ -\frac{b^3}{a^3} - 3.\ \frac{c}{a}\left(-\frac{b}{a}\right)\right\} x - \frac{b^3}{a^3} = 0$.

or $\quad x^2 + \left\{\frac{b^3 - 3abc}{a^3}\right\} x - \frac{b^3}{a^3} = 0$

or $\quad a^3x^2 + (b^3 - 3abc)x - b^3 = 0.$



On the squared paper, take the points A and B 50 miles apart. Draw through A and B the graph of A's and B's motion by the method adopted in Exercise IV. 8. This is done by marking $A1 = 1$ hour and $1C = 8$ miles, and joining AC. This gives the graph of A's motion. Also marking $B1' = 1$ hour, $1'D = 12$ miles, and joining BD, we get that of B's motion.

Let these intersect at O. The time corresponding to this point $= AM = 2\frac{1}{2}$ **hours.**

A's distance from his start $=$ **20 miles** at the instant of meeting.

These give the required answers.

Exercise X.

1. (*a*). If $2s=a+b+c$, prove that
$$(s-a)^2+(s-b)^2+(s-c)^2+s^2=a^2+b^2+c^2$$

(*b*) Prove that
$$4xy(x^2+y^2)=(x^2+xy+y^2)^2-(x^2-xy+y^2)^2$$

2. (*a*) Simplify $\dfrac{x-a}{b}-\dfrac{x-b}{a}$ when $x=a^2$

(*b*) Shew that $n(n-1)(n-2)-p(p-1)(p-2)$
$$=(n-p)\{(n+p-1)(n+p-2)-np\}.$$

Simplify $\dfrac{x^6-y^6}{x^8+x^4y^4+y^8}$ and $\dfrac{x^3+2x^2+2x}{x^5+4x}$.

Solve (*i*) $\sqrt{x}+\sqrt{x-\sqrt{1}}$ -

(*ii*) $$\begin{cases} x-y=3 \\ \dfrac{\dfrac{1}{y}+\dfrac{1}{x}}{\dfrac{1}{y}-\dfrac{1}{x}}=\dfrac{11}{3} \end{cases}$$

(*iii*) $x=\dfrac{x+4}{x-1}$.

5. Find three consecutive numbers whose sum is 33.

6 Two boats start for a race; the second boat rows 25 strokes to the first's 28; but 5 strokes of the 2nd are equal to 6 strokes of the first; if the distance between the boats = 30 strokes of the 2nd boat, after how many strokes will it bump the first?

7. A travels at 5 miles an hour, but takes a rest of half an hour at the end of each hour. B starting 2 hours after A and travelling uniformly, without resting, overtakes A $17\frac{1}{2}$ miles from home. Find graphically B's rate of travelling.

1. (a) $(s-a)^2+(s-b)^2+(s-c)^2+s^2$

$=(s^2-2as+a^2)+(s^2-2bs+b^2)$

$+(s^2-2cs+c^2)+s^2$ [Art. 10. B.

$=4s^2-2as-2bs-2cs+a^2+b^2+c^2$

$=4s^2-2s(a+b+c)+a^2+b^2+c^2$ [Art. 17.

$=4s^2-2s.\ 2s+a^2+b^2+c^2$ [$\because\ a+b+c=2s$. by Hyp.

$=a^2+b^2+c^2.$

(b) $4xy\ (x^2+y^2)=2xy\ (2x^2+2y^2)$

But $2xy=(x^2+xy+y^2)-(x^2-xy+y^2)$

And $2x^2+2y^2=(x^2+xy+y^2)+(x^2-xy+y^2)$

$\therefore$ Product $=(x^2+xy+y^2)^2-(x^2-xy+y^2)^2$ [...

2. (a) $\dfrac{x-a}{b}-\dfrac{x-b}{a}=\dfrac{ax-a^2-bx+b^2}{ab}=\dfrac{x(a-b)-a^2+b^2}{ab}.$

But $x=\dfrac{a^2}{a-b}$ by hypothesis, or

$x(a-b)=a^2$ [Art. 21. C.

Ans. $\dfrac{a^2-a^2+b^2}{ab}=\dfrac{b^2}{ab}=\dfrac{b}{a}$

(b) $\because\ (n-1)\ (n-2)=n^2-3n+2$ [13. D.

and $(p-1)\ (p-2)=p^2-3p+2.$

$\therefore$ the given expression

$=n(n^2-3n+2)-p(p^2-3p+2)$

$n^3-p^3-3n^2+3p^2+2n-2p$

$=(n-p)\ (n^2+np+p^2)-3(n+p)\ (n-p)+2(n-p).$

$=(n-p)\ \{n^2+np+p^2-3(n+p)+2\}$ [Art. 17.

$=(n-p)\ \{n^2+2np+p^2-3(n+p)+2-np\}$

$=(n-p)\ \{(n+p)^2-3(n+p)+2-np\}$ [16. A.

But $\because\ x^2-3x+2=(x-1)\ (x-2)$ [13. D.

$\therefore$ Supposing $x=(n+p)$

The above $=(n-p)\ \{(n+p-1)\ (n+p-2)-np\}.$

3. $1\text{st} = \dfrac{(x^3+y^3)\,(x^3-y^3)}{x^8+2x^4y^4+y^8-x^4y^4}$ [16. C.

$= \dfrac{(x+y)\,(x^2-xy+y^2)\,(x-y)\,(x^2+xy+y^2)}{(x^4+y^4)^2-(x^2y^2)^2}$ [18A. and B.

of which the denominator

$= (x^4+y^4+x^2y^2)\,(x^4+y^4-x^2y^2)$ [16. C.

But $x^4+y^4+x^2y^2 = x^4+2x^2y^2+y^4-x^2y^2$.

$= (x^2+y^2)^2-(xy)^2$ [16. A.

$= (x^2+y^2+xy)\,(x^2+y^2-xy)$ [See Example 2, Exercise IX; also note 2 of Example 2(*iii*) of Exercise VIII.]

$\text{Ans.} = \dfrac{(x+y)\,(x-y)}{x^4-x^2y^2+y^4}$.

$2\text{nd.} = \dfrac{x(x^2+2x+2)}{x(x^4+4)}$

But $x^4+4 = (x^2+2x+2)\,(x^2-2x+2)$ [Ex. IX. 3 (*a*).

$\therefore \text{Ans.} = \dfrac{1}{x^2-2x+2}$.

4. By transposition

$$\sqrt{x-\sqrt{1-x}} = 1-\sqrt{x}$$ [21. A.

Squaring $x-\sqrt{1-x} = 1+x-2\sqrt{x}$ [21. E. Axiom. 10.B.

$\therefore \quad -\sqrt{1-x} = 1-2\sqrt{x}$ [21. A. Axiom.

Again squaring

$$1-x = 1+4x-4\sqrt{x}$$

$$\therefore \quad -5x = -4\sqrt{x}$$

$$\therefore \quad 25x^2 = 16x$$

Dividing by $25x$, we get $x = \dfrac{16}{25}$ [21. D.

(*ii*) In the 2nd Equation, apply Art. 22. E.

$$\frac{\left(\frac{1}{y}+\frac{1}{x}\right)+\left(\frac{1}{y}-\frac{1}{x}\right)}{\left(\frac{1}{y}+\frac{1}{x}\right)-\left(\frac{1}{y}-\frac{1}{x}\right)} = \frac{11+3}{11-3} \text{ or } \frac{14}{8}.$$

$$\frac{\frac{1}{y}}{\frac{1}{x}} \text{ or } \frac{x}{y} = \frac{7}{4} \quad \therefore \ x = \frac{7}{4}y.$$

Substitute this value in the first equation.

$$\frac{7}{4}y - y = 3, \text{ or } \frac{3y}{4} = 3 \quad \therefore \ y = 4\ ; \ \therefore \ x = 7.$$

(*iii*) $x = \dfrac{x+4}{x-1}$

$\therefore \ x(x-1) = x+4$ [Art. 21. C.

or $x^2 - x = x + 4$.

or $x^2 - 2x - 4 = 0$

$\therefore$ By Art. 23. A, $x = \dfrac{2 \pm \sqrt{4+16}}{2} = \dfrac{2 \pm \sqrt{20}}{2} = 1 \pm \sqrt{5}$

5. Let $x-1$, x and $x+1$ be the three numbers.

$\therefore \quad x - 1 + x + x + 1 = 33$ [Hyp.

$\therefore \quad 3x = 33$

$\therefore \quad x = 11.$

$\therefore \quad x - 1 = 10$; and $x + 1 = 12$.

6. Let x = number of strokes made by the *2nd* boat, and y = space over which it is impelled by 1 stroke.

$\therefore \quad 30y$ = distance between the boats.

Now 6 strokes of the first $= 5$ strokes of the *2nd* $= 5y$.

$\therefore$ 1 stroke of the first $= \frac{5y}{6}$;

Again, when the *2nd* makes 25 strokes, the first makes 28.

$\therefore$ when the *2nd* makes x strokes, the first makes $\frac{28x}{25}$.

Now, by the question, the *2nd* boat rows over the space of $30y$ more than the *1st*.

$$\therefore \quad \frac{28x}{25} \times \frac{5y}{6} + 30y = xy.$$

Rejecting y which is common.

$$\frac{14x}{15} + 30 = x$$

$$\therefore \quad x = 450.$$

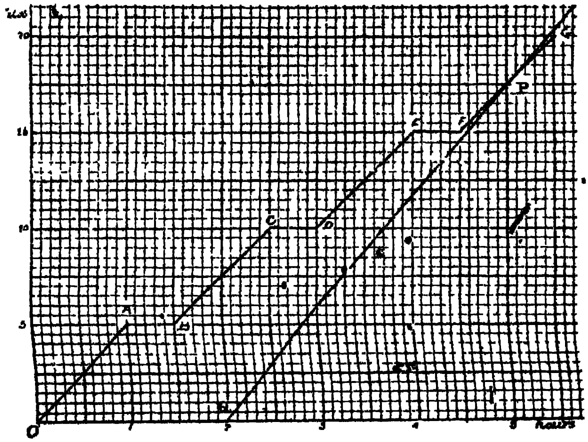
7. Measure time horizontally, and miles vertically as shown in the figure.

OA is A's graph of motion for the first hour For the next half hour he rests. .. AB is his graph for that time. In the same way BC is his graph for the next hour and CD that of his rest and so on.

B starts 2 hours after A. Initially, therefore his position on the graph is Q. He overtakes A at the point P such that P is on the graph of A's motion corresponding to a distance of $17\frac{1}{2}$ miles from home.

Since B travels uniformly without resting, QP is his graph.

From the figure we can easily read off his speed which is $5\frac{3}{8}$ miles per hour.



Exercise XI.

1. (*a*) If two square numbers be added together, the double of the result is also the sum of two square numbers.

(*b*) Find the coefficient of x in the expansion of

$$(2x+3)\,(3x+4)\,(4x-5).$$

2. (*i*) If $x^y = y^x$, show that $\left(\frac{x}{y}\right)^{x} = y^{\frac{y}{x}+1}$.

(*ii*) Resolve into elementary factors

$$m^4 - n^4 + 2n\,(m^3 + n^3) - (m+n)^2\,(m-n)^2.$$

3. Reduce to its lowest terms $\dfrac{x^4 - x^3 - x + 1}{x^4 - 2x^3 - x^2 - 2x + 1}$.

4. Simplify $\dfrac{1}{a(a-b)(a-c)} + \dfrac{1}{b(b-c)(b-a)} + \dfrac{1}{c(c-a)(c-b)}$.

5. Solve (*i*) $\dfrac{1-\sqrt{1-x}}{1+\sqrt{1-x}}=n$

(*ii*) $xy=6$; $xz=8$; $yz=12$.

6. The ten's digit of a number is less by 2 than the unit's digit, and if the digits are inverted the new number is to the former as 7 : 4. Find the digits.

7. If α, β be the roots of the equation

$x^2-px+q=0$, prove that

$\alpha^5+\beta^5=p^5-5p^3q+5pq^2$.

1. (*a*) Let x^2 and y^2 be two square numbers.

Then double their sum $=2(x^2+y^2)=2x^2+2y^2$.

$=(x^2+2xy+y^2)+(x^2-2xy+y^2)$

$=(x+y)^2+(x-y)^2$ [16. A & B.

(*b*) $2x+3=2.\ x+2.\ \frac{3}{2}=2(x+\frac{3}{2})$ [Art. 17.

$3x+4=3.\ x+3.\ \frac{4}{3}=3(x+\frac{4}{3})$ [Art. 17.

$4x-5=4.\ x-4.\ \frac{5}{4}=4(x-\frac{5}{4})$ [Art. 17.

Their product $=2\times3\times4\times(x+\frac{3}{2})\ (x+\frac{4}{3})\ (x-\frac{5}{4})$

$=24\times\{x^3+x^2(\frac{3}{2}+\frac{4}{3}-\frac{5}{4})+x(\frac{3}{2}\cdot\frac{4}{3}-\frac{3}{2}\cdot\frac{5}{4}-\frac{4}{3}\cdot\frac{5}{4})-\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\}$

[Art. 14. B.

$\therefore$ Coefficient of $x=24(\frac{3}{2}\cdot\frac{4}{3}-\frac{3}{2}\cdot\frac{5}{4}-\frac{4}{3}\cdot\frac{5}{4})$.

$=24(2-\frac{15}{8}-\frac{5}{3})=48-45-40=-37$.

2. (*i*) $\left(\frac{x}{y}\right)^{-\frac{y}{x}} = \left(\frac{y}{x}\right)^{\frac{y}{x}}$ [Art. 8.

$= y^{\frac{y}{x}} \div x^{\frac{y}{x}}$ .....1 But $x^{y} = y^{-x}$ [Hyp.

Extracting x^{th} root, $x^{\frac{y}{x}} = y^{\frac{-x}{x}} = y^{-1}$.

$\therefore$ 1 $= y^{\frac{y}{x}} \div y^{-1} = y^{\frac{y}{x}-(-1)}$ [Art. 8.

$= y^{\frac{y}{x}+1}$.

$(ii) = (m^2+n^2)(m^2-n^2) + 2n(m^3+n^3) - (m^2-n^2)^2$.

$= (m^2-n^2)\{m^2+n^2-(m^2-n^2)\} + 2n(m^3+n^3)$

$= 2n^2(m^2-n^2) + 2n(m+n)(m^2-mn+n^2)$ [17 A.

$= 2n(m+n)\{n(m-n)+m^2-mn+n^2\}$

$= 2n(m+n)\{mn-n^2+m^2-mn+n^2\}$

$= 2n(m+n)m^2$. [Arts. 16. C, 17, 18 A.

3. Numr. $= x^3(x-1) - 1(x-1) = (x^3-1)(x-1)$ [Art. 17.

$= (x-1)(x^2+x+1)(x-1)$ [Art. 18. B.

Denmr. $= x^4 + x^2 + 1 - 2x^3 - 2x^2 - 2x$

$= (x^2+x+1)(x^2-x+1) - 2x(x^2+x+1)$ [Ex. VIII. 2(iii)

$= (x^2+x+1)(x^2-x+1-2x)$

$= (x^2+x+1)(x^2-3x+1)$

$\therefore$ Ans. $= \dfrac{x^2-2x+1}{x^2-3x+1}$ $\because (x-1)(x-1) = x^2-2x+1$ by 10. B.

4\. See Ex. VII, Example 3(*b*).

$$\frac{1}{a(a-b)(a-c)}-\frac{1}{b(b-c)(a-b)}+\frac{1}{c(a-c)(b-c)}$$

$$=\frac{bc(b-c)-ac(a-c)+ab(a-b)}{abc(a-b)(a-c)(b-c)}$$

Of which the Numerator

$= ab(a-b) - a^2c + b^2c + ac^2 - bc^2$

$= ab(a-b) - c(a^2-b^2) + c^2(a-b)$ which by Art. 17

$= (a-b)\{ab - c(a+b) + c^2\}$ which by 13. D. Conv.

$= (a-b)(a-c)(b-c)$.

$\therefore$ Ans $= \dfrac{1}{abc}$.

5\. (*i*) By Art. 22 A

$$\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}=\frac{1}{n} \quad \therefore \quad \frac{1}{\sqrt{1-x}}=\frac{1+n}{1-n}$$ [22. E.

Squaring $\dfrac{1}{1-x}=\left[\dfrac{1+n}{1-n}\right]^2$ [21. E.

$1-x=\left[\dfrac{1-n}{1+n}\right]^2$ [22. A.

$$x=1-\left[\frac{1-n}{1+n}\right]^2=\frac{4n}{(1+n)^2}.$$

(*ii*) $xy \times xz \times yz$ or $x^2y^2z^2 = 6 \times 8 \times 12$

[By multiplying the three equations.

$\therefore$ $xyz = \sqrt{24 \times 24} = 24 \ldots\ldots 1.$

Divide (1) by the 3rd equation,

$\therefore$ $\dfrac{xyz}{yz}$ or $x = 2$. Similarly $\dfrac{xyz}{xz}$ or $y = \dfrac{24}{8}$ or 3

and $\dfrac{xyz}{xy}$ or $z = \dfrac{24}{6}$ or 4.

6. Let x = ten's digit, and y = unit's digit.

$\therefore$ By the question, $x+2=y$......1

Now, x and y being the digits, the number $=10x+y$.

But if the digits be inverted, the new number $=10y+x$. [Art. 1.

$$\therefore \quad \frac{10y+x}{10x+y}=\frac{7}{4} \text{ ....} \mathbf{2}$$

$\therefore$ $40y+4x=70x+7y$; $33y=66x$ or $y=2x$.

Substitute this value of y in **1**

$\therefore$ $x+2=2x$ or $x=2$ (Ten's digit)

$\therefore$ $y=4$. (Unit's digit)

7. By Art 25. $\alpha+\beta=p$; $\alpha\beta=q$.

By actual multiplication,

$$(\alpha+\beta)^5=\alpha^5+5\alpha^4\beta+10\alpha^3\beta^2+10\alpha^2\beta^3+5\alpha\beta^4+\beta^5$$

$$=\alpha^5+\beta^5+5\{\alpha\beta(\alpha^3+\beta^3)\}+10\alpha^2\beta^2(\alpha+\beta)$$

$$=\alpha^5+\beta^5+5\{\alpha\beta[(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)]\}+10\alpha^2\beta^2(\alpha+\beta).$$

Substituting the above values.

$$p^5=\alpha^5+\beta^5+5\{q[p^3-3pq]\}+10q^2p.$$

$$\therefore \quad \alpha^5+\beta^5=p^5-5\{p^3q+3pq^2\}-10q^2p$$

$$=p^5-5p^3q+15pq^2-10q^2p$$

$$=p^5-5p^3q+5pq^2.$$

Exercise XII.

1. Show that the sum of the squares of any two numbers is greater than twice the product of the numbers.

2. (a) Show that
$(ay-bx)^2+(bz-cy)^2+(cx-az)^2+(ax+by+cz)^2$ is divisible by $a^2+b^2+c^2$ and by $x^2+y^2+z^2$.

(b) Find the G. C. M. of
$$x^3+2x^2y+2xy^2+y^3\text{ ; and }x^4-x^3y+xy^3-y^4.$$

3. Simplify

(a) $$\frac{x^3}{x-1}-\frac{x^2}{x+1}-\frac{1}{x-1}+\frac{1}{x+1}.$$

(b) $$\left\{\frac{a^{\frac{x}{x-y}}\sqrt[y]{b^{-(y-x)}}}{\sqrt[y]{a^{x+y}\,b^{x}}\;a^{\frac{y}{x-y}}}\right\}^{-y}$$

4. Solve (i) $$\frac{x+4a+b}{x+a+b}+\frac{4x+a+2b}{x+a-b}=5.$$

(ii) $a(x+y)=b(x-y)=xy.$

5. A cask A contains 12 gallons of wine and 18 gallons of water; and another cask B contains 9 gallons of wine and three gallons of water. How many gallons must be drawn from each cask so as to produce by their mixture 7 gallons of wine and 7 gallons of water.

6. Find three consecutive even numbers the sum of whose squares is 2036.

7. If α, β are the roots of $x^2+px-q=0$, and γ, δ those of $x^2+px+r=0$, prove that
$$(\alpha-\gamma)(\alpha-\delta)=(\beta-\gamma)(\beta-\delta)=q+r.$$

1. Let x and y be two numbers.

$\because$ the square of a number, be it positive or negative, is positive, and all positive quantities are by definition greater than zero.

$\therefore$ the square of any number is greater than zero.

Hence, $(x-y)^2>0$; *i. e.*, $x^2-2xy+y^2>0$ [Art. 10. B

Add $2xy$ to both sides, $\therefore$ $x^2+y^2>2xy$ [21. B. Axiom.

2. (*a*)
$$\left.\begin{aligned}(ay-bx)^2&=a^2y^2+b^2x^2-2abxy\\(bz-cy)^2&=b^2z^2+c^2y^2-2bcyz\\(cx-az)^2&=c^2x^2+a^2z^2-2acxz\end{aligned}\right\}\quad[\text{ 10. B.}$$

$$(ax+by+cz)^2=a^2x^2+b^2y^2+c^2z^2+2abxy+2acxz+2bcyz$$

[10 C.

Adding these we get.

$$a^2x^2+b^2x^2+c^2x^2+a^2y^2+b^2y^2+c^2y^2+a^2z^2+b^2z^2+c^2z^2$$
$$=x^2(a^2+b^2+c^2)+y^2(a^2+b^2+c^2)+z^2(a^2+b^2+c^2)\quad[\text{ 17.}$$
$$=(a^2+b^2+c^2)\,(x^2+y^2+z^2)\quad[\text{ 17.}$$

$\therefore$ the given expression is divisible by both
$x^2+y^2+z^2$ and $a^2+b^2+c^2$.

(*b*) 1*st* quantity$=x^3+y^3+2x^2y+2xy^2$

$=(x+y)\,(x^2-xy+y^2)+2xy\,(x+y)$ [17 & 18. A.

$=(x+y)\,(x^2-xy+y^2+2xy)$ [17.

$=(x+y)\,(x^2+xy+y^2)$.

2*nd* $=x^3(x-y)+y^3(x-y)=(x-y)\,(x^3+y^3)$ [17.

$=(x-y)\,(x+y)\,(x^2-xy+y^2)$ [18. A.

$\therefore$ $x+y$ is the G. C. M.

3. (*a*) $\because \dfrac{x}{a} - \dfrac{y}{a} = \dfrac{x-y}{a}$. [Identity for the same denominator.

$$\therefore \quad \frac{x^3}{x-1} - \frac{1}{x-1} = \frac{x^3-1}{x-1} = x^2+x+1 \qquad [\ 18.\ \mathrm{B}.$$

$$\text{Again} \quad \frac{1}{x+1} - \frac{x^2}{x+1} = \frac{1-x^2}{x+1} = 1-x \qquad [\ 16.\ \mathrm{C}.$$

$\therefore$ Ans $= x^2+x+1+1-x = x^2+2$.

$$(b)\ \text{Ans.} = \left\{ \frac{a^{\frac{x}{x-y}}\, b^{-\frac{y-x}{y}}}{a^{\frac{x+y}{y}}\, b^{\frac{x}{y}}\, a^{\frac{y}{x-y}}} \right\}^{-y} \qquad [\ \text{Art. } 7.$$

$$= \left\{ \frac{a^{\frac{x}{x-y}}}{a^{\frac{y}{x-y}}} \cdot \frac{1}{a^{\frac{x+y}{y}}} \cdot \frac{b^{\frac{x-y}{y}}}{b^{\frac{x}{y}}} \right\}^{-y} \qquad [\ \text{Art. } 3.$$

$$= \left\{ a^{\frac{x-y}{x-y} - \frac{x+y}{y}} \times b^{\frac{(x-y)-x}{y}} \right\}^{-y} \qquad [\ \text{Art. } 6.$$

$$= \left(a^{1-\frac{x+y}{y}}\, b^{-1} \right)^{-y}$$

$$= \left(a^{-\frac{x}{y}} \times b^{-1} \right)^{-y} = a^{x} b^{y} \qquad [\ \text{Arts } 5\ \&\ 9.$$

4. (*i*) $1+\dfrac{3a}{x+a+b}+4-\dfrac{3a-6b}{x+a-b}=5.$

[Dividing the Numr. by the Denmr.

$\therefore \dfrac{3a}{x+a+b}-\dfrac{3(a-2b)}{x+a-b}=0$ [Subtracting 5 from both sides.

$\therefore \dfrac{a}{x+a+b}=\dfrac{a-2b}{x+a-b}$ [Dividing by 3, and by transposition.

$\dfrac{x+a+b}{x+a-b}=\dfrac{a}{a-2b}$ [Art. 22. A. & B.

$\dfrac{x+a}{b}=\dfrac{2a-2b}{2b}=\dfrac{a-b}{b}$. [Art. 22. E.

$\therefore x+a=a-b$ or $x=-b.$

(*ii*) Dividing the Equations by xy,

$a\left(\dfrac{x}{xy}+\dfrac{y}{xy}\right)=1$ or $a\left(\dfrac{1}{y}+\dfrac{1}{x}\right)=1.$

$b\left(\dfrac{x}{xy}-\dfrac{y}{xy}\right)=1$ or $b\left(\dfrac{1}{y}-\dfrac{1}{x}\right)=1.$

$\dfrac{1}{y}+\dfrac{1}{x}=\dfrac{1}{a}$ ....(A) [Dividing the 1st by a.

$\dfrac{1}{y}-\dfrac{1}{x}=\dfrac{1}{b}$ ....(B) [Dividing the 2nd by b.

$\therefore \dfrac{2}{y}=\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{a+b}{ab}$ [Adding (A) and (B)

$\therefore y=\dfrac{2ab}{a+b}$ [Art. 22. A & 21. C.

Again, $\dfrac{2}{x}=\dfrac{1}{a}-\dfrac{1}{b}=\dfrac{b-a}{ab}$ [Subtracting (B) from (A)

$\therefore x=\dfrac{2ab}{b-a}=-\dfrac{2ab}{a-b}$ [Ex. VII—3(*b*)

5. Let x=No. of gallons to be drawn from the cask A.

$\because$ 14 gallons of mixture is required,

$\therefore$ $14-x$= No. of gallons to be drawn from the cask B.

$\because$ In 30 gallons of mixture in A, the quantity of wine = 12 gallons.

$\therefore$ In x gallons of mixture in A, the quantity of wine $=\frac{12x}{30}=\frac{2x}{5}$.

Similarly, in $14-x$ gallons of mixture in B, the quantity of wine $=\frac{9}{12}(14-x)=\frac{3}{4}(14-x)$

$$\therefore \quad \frac{2x}{5}+\frac{3}{4}(14-x)=7.$$

or $8x+(42-3x)5=140$

or $8x+210-15x=140$

or $7x=70 \quad \therefore \quad x=10.$

6. $\because$ the difference of two consecutive even numbers is 2,

Let $x-2$, x, and $x+2$ be the numbers required.

$\therefore \quad (x-2)^2+x^2+(x+2)^2=2036$ [Hyp.

$x^2-4x+4+x^2+x^2+4x+4=2036$ [10 A & B

$\therefore \quad 3x^2+8=2036\ ; \ \therefore 3x^2=2028 \quad \therefore x^2=676.$

$\therefore \quad x=\sqrt{676}=26\ ; \ \therefore x-2=24$ and $x+2=28$

7. By art. 25, $\alpha+\beta=-p \qquad \gamma+\delta=-p$

$\alpha\beta=-q \qquad \gamma\delta=r$

Now $(\alpha-\gamma)(\alpha-\delta)$

$=\alpha^2-\alpha(\gamma+\delta)+\gamma\delta$ [Art. 13. D.

$=\alpha^2+p\alpha+r$

$=\alpha(\alpha+p)+r.$

Now $\because\ \alpha+\beta=-p \quad \therefore\ \alpha+p=-\beta.$

$\therefore\ \alpha(\alpha+p)=-\alpha\beta=-(-q)=q.$

$\therefore\ (\alpha-\gamma)(\alpha-\delta)=q+r.$

Similarly $(\beta-\gamma)(\beta-\delta)$

$=\beta^2-(\gamma+\delta)\beta+\gamma\delta$ [Art. 13 D.

$=\beta^2+p\beta+r$

$=\beta(\beta+p)+r$ [$\because\ \alpha+\beta=-p\ \therefore\ \beta+p=-\alpha$

$=-\alpha\beta+r$

$=q+r.$

$\therefore\ (\alpha-\gamma)(\alpha-\delta)=(\beta-\gamma)(\beta-\delta)=q+r.$

Exercise XIII.

1 Find the condition that x^2+ax+b^2 may be a multiple of $x+c$, and divide $x^{2^n}-y^{2^n}$ by $x^{2^{n-1}}-y^{2^{n-1}}$.

2 If $S=\dfrac{a+b+c}{2}$, show that

$$\left(\frac{a^2+b^2-c^2}{2b}\right)^2 \quad 4s(s-a)(s-b)(s-c)$$

3 Simplify (a) $\dfrac{a^5-a^4b-ab^4+b^5}{a^4-ba^3-a^2b^2+ab^3}.$

$$(b)\left\{2\sqrt{a^{\frac{1}{3}}b^{-\frac{1}{3}}}+3\sqrt[3]{a^{\frac{1}{2}}\sqrt{b^{-1}}}-4\sqrt[4]{ab^{-1}}\ \sqrt[3]{ab^{-1}}\right\}$$

4. Solve (a) $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}=\dfrac{x-5}{(x-6)}-\dfrac{x-6}{x-7}.$

(b) $a(x+y)=b(x+z)=c(y+z)=1.$

5. At what time between one o'clock and two o'clock is there exactly one minute division between the two hands of a clock?

6 The sum of Rs 21 4 as is divided among ten boys and a certain number of girls in such a way that the number of annas each boy receives, bears to the number of pies each girl receives, the ratio of 3 to 5 How many girls are there, the share of each being $3\frac{1}{3}$ annas?

7 In the ten years from 1881 to 1890, the population of one town increases uniformly from 30,000 to 50,000 whilst that of another town decreases from 60,000 to 40,000. From a graph determine the year and month when the two populations were equal.

1 (a) x^2+ax+b^2 will be a multiple of $x+c$ if being divided by $x+c$, it leave no remainder. From Art. 20, we see that there is a remainder $(-c)^2+(a)\times(-c)+b^2$,

or b^2+c^2-ac. [Art. 3

$\therefore$ The required condition is that

$$b^2+c^2-ac=0$$

or $b^2+c^2=ac$ [Art 21. B

(b) $x^{2^n}-y^{2^n}=x^{\frac{2^n}{2}\cdot 2}-y^{\frac{2^n}{2}\cdot 2}$

$=\left(x^{\frac{2^n}{2}}\right)^2-\left(y^{\frac{2^n}{2}}\right)^2$. [5 Conv.

$=\left(x^{\frac{2^n}{2}}+y^{\frac{2^n}{2}}\right)\left(x^{\frac{2^n}{2}}-y^{\frac{2^n}{2}}\right)$ [16. C.

$=\left(x^{2^{n-1}}+y^{2^{n-1}}\right)\left(x^{2^{n-1}}-y^{2^{n-1}}\right)$ $\left[\because \frac{2^n}{2}=2^{n-1}\right.$ by Art. 6.

$\therefore$ Ans $=x^{2^{n-1}}+y^{2^{n-1}}$.

2 $$\left(a+\frac{a^2+b^2-c^2}{2b}\right)\times\left(a-\frac{a^2+b^2-c^2}{2b}\right) \qquad [\ 16.\ C.$$

$$=\left(\frac{2ab+a^2+b^2-c^2}{2b}\right)\times\left(\frac{2ab-a^2-b^2+c^2}{2b}\right) \qquad [\ \text{Art. } 3.$$

$$=\frac{(a+b)^2-c^2}{2b}\times\frac{c^2-(a-b)^2}{2b}$$

$$=\frac{(a+b+c)(a+b-c)}{2b}\times\frac{(c+a-b)(c-a+b)}{2b}.$$

$$=\frac{2s(2s-2c)}{2b}\times\frac{(2s-2b)(2s-2a)}{2b}$$

[∵ By Hyp. $a+b+c=2s$, and
$a+b-c=a+b+c-2c=2s-2c$ and so on].

$$=\frac{4s(s-a)(s-b)(s-c)}{b^2}.$$

3 (a) $$\frac{a^4(a-b)-b^4(a-b)}{a^3(a-b)-ab^2(a-b)}=\frac{(a^4-b^4)(a-b)}{(a^3-ab^2)(a-b)}. \quad [\ \text{Art. } 17$$

$$=\frac{(a^2+b^2)(a^2-b^2)}{a(a^2-b^2)}=\frac{a^2+b^2}{a}.$$

(b) $$\begin{cases} 2\,(a^{\frac{1}{2}}b^{-\frac{1}{2}})^{\frac{1}{3}}=2a^{\frac{1}{6}}b^{-\frac{1}{6}} \quad \ldots\ldots\ldots\ldots 1 \\ 3\,(a^{\frac{1}{2}}b^{-\frac{1}{2}})^{\frac{1}{3}}=3a^{\frac{1}{6}}b^{-\frac{1}{6}} \quad \ldots\ldots\ldots\ldots 2 \end{cases} \qquad [\ \text{Art. } 5.$$

$$ab^{-1}\times(ab^{-1})^{\frac{1}{3}}=(ab^{-1})^{1+\frac{1}{3}}=(ab^{-1})^{\frac{4}{3}}=a^{\frac{4}{3}}b^{-\frac{4}{3}}. \qquad [\ \text{Art. } 4.$$

Raising this to $\frac{1}{8}$th the power, $a^{\frac{4}{3}\times\frac{1}{8}}b^{-\frac{4}{3}\times\frac{1}{8}}=a^{\frac{1}{6}}b^{-\frac{1}{6}}$.

Multiplying this by 4, we get $4a^{\frac{1}{6}}b^{-\frac{1}{6}}$ **3**

Adding **1** and **2** and subtracting **3** from the sum, we get

$$5a^{\frac{1}{6}}b^{-\frac{1}{6}}-4a^{\frac{1}{6}}b^{-\frac{1}{6}}=a^{\frac{1}{6}}b^{-\frac{1}{6}}.$$

Raising this to the sixth power, we get the answer, which

$$\left(a^{\frac{1}{6}}b^{-\frac{1}{6}}\right)^6=ab^{-1}=\frac{a}{b}.$$

4. (*a*) Divide, as in example 4 (*i*) Exercise XII, the Numerator by the Denominator.

$$1+\frac{1}{x-2}-\left(1+\frac{1}{x-3}\right)=\frac{1}{x-6}+1-\left(1+\frac{1}{x-7}\right)$$

$$\text{or}\quad \frac{1}{x-2}-\frac{1}{x-3}=\frac{1}{x-6}-\frac{1}{x-7}$$

$$\therefore\quad \frac{-1}{(x-2)(x-3)}=\frac{-1}{(x-6)(x-7)}$$

$$\therefore\quad (x-6)(x-7)=(x-2)(x-3)$$

$$\therefore\quad x^2-13x+42=x^2-5x+6 \qquad [13.\ C.$$

$$\therefore\quad 8x=36 \text{ or, } x=4\tfrac{1}{2}.$$

4. (*b*) Dividing the equations by a, b and c respectively,

$$x+y=\frac{1}{a}\ \ldots\ldots\ldots\ldots\ldots\ldots 1$$

$$x+z=\frac{1}{b}\ \ldots\ldots\ldots\ldots\ldots\ldots 2$$

$$y+z=\frac{1}{c}\ \ldots\ldots\ldots\ldots\ldots\ldots 3$$

$$\therefore\ 2(x+y+z)=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \qquad \text{[Adding 1, 2 and 3.}$$

$$\text{or, } x+y+z=\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}\ \ldots\ldots\ldots 4$$

$$\therefore\quad x=\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}-\frac{1}{c} \qquad \text{[Subtracting 3 from 4}$$

$$=\frac{1}{2a}+\frac{1}{2b}-\frac{1}{2c}$$

$$\text{Similarly } y=\frac{1}{2a}-\frac{1}{2b}+\frac{1}{2c} \qquad \text{[Subtracting 2 from 4}$$

$$\text{and } z=\frac{1}{2b}+\frac{1}{2c}-\frac{1}{2a} \qquad \text{[Subtracting 1 from 4}$$

5. In one hour, the hour-hand moves over the space of 5 minutes, and the minute-hand moves over that of 60 minutes.

$\therefore$ The minute-hand travels 12 times as fast as the hour hand.

Just at one o'clock the space between the two hands was 5 minutes.

Let x = No. of minutes past one o'clock, when there is a division of one minute between the two hands.

But when the minute hand moves over x minutes, the hour hand moves over $\frac{x}{12}$ minutes.

$\therefore \quad x = 5 + \frac{x}{12} \mp 1$ ($-$ or $+$ according as the minute-hand goes before or after the hour-hand).

$$\therefore \quad x - \frac{x}{12} = 5 \mp 1 \qquad \text{or} \quad \frac{11x}{12} = 4 \text{ or } 6.$$

$$\therefore \quad x = \frac{48}{11} \text{ or } \frac{72}{11} = 4\frac{4}{11} \text{ or } 6\frac{6}{11} \text{ minutes.}$$

6. Let x=No. of girls required.

The total amount received by them in annas$=3\frac{1}{3}\,x$.

$\therefore$ The sum distributed among the boys

$= \text{Rs. } 21.\ 4 \text{ as} - 3\frac{1}{3}x \text{ annas.}$

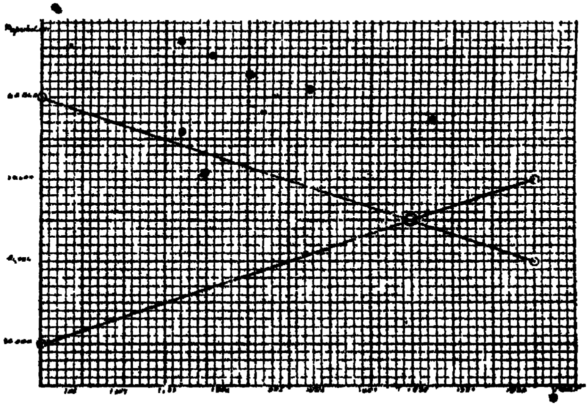
$$= 340 - \frac{10x}{3} \text{ annas}$$

Hence, $\left(340 - \frac{10x}{3}\right) \div 10 : 40 :: 3 : 5$ [Hyp.

$$i.e.\ 340 - \frac{10x}{3} = 240$$

$$\therefore \quad \frac{10x}{3} = 100 \quad \therefore \quad x = 30.$$

7. Mark off the years on the horizontal line and the population on the vertical line. The figure explains how the increasing and decreasing populations are represented by two straight lines. The intersection of these lines gives the year and month as June 1888.



Exercise XIV.

1. If $s=a+b+c$, prove that $s(s-2b)(s-2c)+s(s-2c)(s-2a)+s(s-2a)(s-2b)=(s-2a)(s-2b)(s-2c)+8abc$.

2. Find for what value of x the expression

$x^4+6x^3+11x^2+3x+31$ is a perfect square.

3. Simplify $\dfrac{a^3+b^3+c^3-3abc}{(a-b)^2+(b-c)^2+(c-a)^2}$.

4. Solve the equations—

(a) $\left(\dfrac{x-a}{x+b}\right)^3 = \dfrac{x-2a-b}{x+a+2b}$.

(b) $x^2yz=6$; $xy^2z=12$; $xyz^2=18$.

5. If $\dfrac{bx^2+a^2y}{a}=\dfrac{ay^2+b^2x}{b}$, prove that

$bx+ay=ab$, or $ay=bx$.

6. A hare is eighty of her own leaps before a greyhound. She takes three leaps for every two that he takes, but he covers as much ground in one leap as she does in two. How many leaps will the hare have taken before she is caught?

7. Construct an equation whose roots shall exceed by a quantity m the roots of the equation $ax^2+bx+c=0$.

1. $s=s-2a+2a$

$\therefore\ s(s-2b)(s-2c)=(s-2a+2a)(s-2b)(s-2c)$

$=(s-2a)(s-2b)(s-2c)+2a(s-2b)(s-2c)$ [Conv. 17.

The whole expression

$=(s-2a)(s-2b)(s-2c)+2a(s-2b)(s-2c)+s(s-2c)$
$(s-2a)+s(s-2a)(s-2b)$

But $2a(s-2b)(s-2c)=2a\{s^2-2s(b+c)+4bc\}$......1 [13 C.

and $s(s-2c)(s-2a)+s(s-2a)(s-2b)$ which by Art 17.

$=s(s-2a)\{s-2c+s-2b\}=s(s-2a)(2s-2b-2c)$

$=s(s-2a)\times 2a=2a(s^2-2as)$....................2.

Adding 1 and 2

$2a\{s^2-2s(b+c)+4bc+s^2-2as\}$

$=2a\{2s^2-2s(a+b+c)+4abc\}$

$=2a\{2s^2-2s^2+4bc\}=8abc$ [$\because\ a+b+c=s$ by Hyp.

$\therefore$ The given expression $=(s-2a)(s-2b)(s-2c)+8abc$,

2. In order that the given quantity may be an exact square there must be no remainder left when the process of squaring has been gone through.

$$x^4+6x^3+11x^2+3x+31\left(x^2+3x+1\right.$$

$$x^4$$

$$2x^2+3x \qquad 6x^3+11x^2$$

$$6x^3+9x^2$$

$$2x^2+6x+1 \qquad 2x^2+3x+31$$

$$2x^2+6x+1$$

$$-3x+30$$

Thus after performing the operation of squaring we see that there is a remainder left. *viz.*, $-3x+30$.

$$\therefore\ -3x+30=0 \quad \text{or } x=10$$

3. $\text{Numr.}=(a+b)^3+c^3-3ab(a+b)-3abc$

$$=(a+b+c)\{(a+b)^2-(a+b)c+c^2\}$$

$$-3ab(a+b+c) \qquad [\text{Art. 17 and 18A.}$$

$$=(a+b+c)\{a^2+2ab+b^2-ac-bc+c^2-3ab\}$$

$$=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$$

Note. 1. This result should be carefully remembered as the factorization of "sum of three cubes *minus* thrice the product" formula—.

Note 2 By changing the letters a, b, c into $(-a)$, $(-b)$ and $(-c)$ severally or jointly, the factors would undergo material change For instance, $a^3+(-b)^3+c^3-3a.(-b).c$ (which $=a^3-b^3+c^3+3abc)=(a-b+c)(a^2+b^2+c^2+ab-ac+bc)$ and, $a^3+(-b)^3+(-c)^3-3a(-b)(-c)$ (which $=a^3-b^3-c^3-3abc)=(a-b-c)(a^2+b^2+c^2+ab+ac-bc)$; and so on.

$$\text{Denmr.}=a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ac+a^2$$

$$=2(a^2+b^2+c^2-ab-ac-bc).$$

$$\therefore\ \text{Ans.}=\frac{a+b+c}{2}.$$

4. (*a*) Apply article 22. C.

$$\frac{(x-a)^3+(x+b)^3}{(x+b)^3}=\frac{x-2a-b+x+a+2b}{x+a+2b}$$

But $(x-a)^3+(x+b)^3=\{x-a+x+b\}\times$

$\{(x-a)^2-(x-a)(x+b)+(x+b)^2\}$ [18. A.

$$\therefore\ \frac{(2x-a+b)\{(x-a)^2-(x-a)(x+b)+(x+b)^2\}}{(x+b)^3}$$

$$=\frac{2x-a+b}{x+a+2b}$$

Now, $2x+b-a$ being a common factor of both sides of the equation may be rejected, and therefore $=0$.

[For, mark that if $ax=bx$, $x(a-b)=0$. But the product of two quantities cannot be equal to 0, unless one or both of the factors $=0$. But $a-b$, being the difference of known different quantities is not $=0$, $\therefore$ x must be $=0$].

$\therefore\ 2x-a+b=0$

$\therefore\ 2x=a-b$; or $x=\dfrac{a-b}{2}$.

(*b*) Similar to Exercise XI – example 5 (ii).

Multiply the three equations

$x^4y^4z^4=6\times12\times18=6\times6\times6\times6.$

$\therefore\ xyz=6$.......................**1**

Dividing the first equation by **1**, $x=1$.

Similarly, $y=2$ [Dividing the 2nd equation by **1**

and $z=3$ [Dividing the 3rd equation by **2**

5. $b^2x^2+a^2by=a^2y^2+ab^2x$ [21. C.

$\therefore\ b^2x^2-a^2y^2-ab^2x+a^2by=0$ [By transposition 21.A.&B.

$(bx+ay)(bx-ay)-ab(bx-ay)=0$ [16. C. and 17.

$(bx-ay)\{bx+ay-ab\}=0.$

$\therefore$ Either $bx-ay=0$ or $bx+ay-ab=0.$

[Read the explanation of the solution No. 4 of this Exercise.]

$\therefore\ bx=ay$; or $bx+ay=ab.$ [21. B.

6. Since the greyhound takes 2 leaps, when the hare does 3 leaps, let us suppose $3x$ = No. of leaps taken by the hare before she is caught.

$\therefore$ $2x$ = No. of leaps taken by the greyhound.

Again, $\because$ when each of the hare's leaps covers a certain portion of ground, each of the greyhound's leaps covers twice that portion.

$\therefore$ $2x$ leaps of the grey hound

$= 4x$ leaps of the hare.

$$80+3x=4x \quad \therefore \quad x=80$$

$\therefore$ $3x=240$ (number of leaps required).

7. Let α and β be the roots of the equation $ax^2+bx+c=0$

$\therefore$ $\alpha+\beta=-\dfrac{b}{a}$; and $\alpha\beta=\dfrac{c}{a}$ [Art. 25.

The roots of the required equation are $\alpha+m$ and $\beta+m$. [By hypothesis.

$\therefore$ The required equation is by Art. 26

$$x^2-x\{\alpha+m+\beta+m\}+(\alpha+m)(\beta+m)=0$$

or $x^2-x\{\alpha+\beta+2m\}+\{\alpha\beta+m(\alpha+\beta)+m^2\}=0$

or $x^2-x\left\{2m-\dfrac{b}{a}\right\}+\left\{\dfrac{c}{a}-\dfrac{b}{a}m+m^2\right\}=0$

or $x^2-x\dfrac{(2am-b)}{a}+\dfrac{c-bm+am^2}{a}=0$

or $ax^2-x(2am-b)+c-bm+am^2=0$

Exercise XV.

1. Prove that

$$\{(a-b)^2+(b-c)^2+(c-a)^2\}^2$$
$$=2\{(a-b)^4+(b-c)^4+(c-a)^4\}$$

2. (a) Divide $z^2+1+\frac{1}{z^2}$ by $z-1+\frac{1}{z}$.

(b) x^2-y^{-2} by $x^{\frac{1}{2}}+y^{-\frac{1}{2}}$.

3. Simplify (a) $\dfrac{x^{3}+3y^{3}}{x^{\frac{1}{3}}-3y^{\frac{1}{3}}}+\dfrac{x^{\frac{2}{3}}-3x^{\frac{1}{3}}y^{\frac{1}{3}}+9y^{\frac{2}{3}}}{x^{\frac{2}{3}}+3x^{\frac{1}{3}}y^{\frac{1}{3}}+9y^{\frac{2}{3}}}$

(b) $\dfrac{x^{2n}}{x^n-1}-\dfrac{x^{2n}}{x^n+1}-\dfrac{1}{x^n-1}+\dfrac{1}{x^n+1}$

4. Solve (a) $\dfrac{\sqrt{a}+\sqrt{a-x}}{\sqrt{a}-\sqrt{a-x}}=\dfrac{1}{a}$.

(b) $\begin{cases} 3x+4y=2xy \\ 4y+3z=3yz \\ 6z+5x=4xz \end{cases}$

5. The sum of two numbers is 4225, and their greatest common measure is 825; show that there are two pairs of numbers satisfying these conditions, and find them.

6. A mixture of a certain quantity of brandy with 20 gallons of water is worth 25 shillings per gallon. If the brandy be worth 30 shillings per gallon, how much brandy is there in the mixture?

7. Find graphically the value of $\sqrt{3}$.

1. Let $\left.\begin{array}{r} a-b=x \\ b-c=y \\ c-a=z \end{array}\right\}$ Adding these together $x+y+z=0$

$\therefore \quad x^2+y^2+z^2+2xy+2xz+2yz=0$ [10. C. & 21. E.

$\therefore \quad x^2+y^2+z^2=-2xy-2xz-2yz$ [21 A.

Squaring both sides, by 10 C

$x^4+y^4+z^4+2x^2y^2+2x^2z^2+2y^2z^2$

$=4x^2y^2+4x^2z^2+4y^2z^2+8x^2yz+8xy^2z+8xyz^2$

$=4x^2y^2+4x^2z^2+4y^2z^2+8xyz(x+y+z)$ [Art. 17.

$\therefore \quad x^4+y^4+z^4=2x^2y^2+2x^2z^2+2y^2z^2$

[$\because x+y+z=0$; and $8xyz\times 0=0$

Adding $x^4+y^4+z^4$ to both sides

$2(x^4+y^4+z^4)=x^4+y^4+z^4+2x^2y^2+2x^2z^2+2y^2z^2$

$=(x^2+y^2+z^2)^2$ [10 C. Conv.

or $\{(a-b)^2+(b-c)^2+(c-a)^2\}^2$

$=2\{(a-b)^4+(b-c)^4+(c-a)^4\}$

[By substituting the values of x, y & z.

2. (*a*) $z^2+1+\dfrac{1}{z^2}=z^2+2+\dfrac{1}{z^2}-1$

$=z^2+2\cdot z.\dfrac{1}{z}+\dfrac{1}{z^2}-1.$ [$\because z\times\dfrac{1}{z}=1.$

$=\left(z+\dfrac{1}{z}\right)^2-1=\left(z+\dfrac{1}{z}+1\right)\left(z+\dfrac{1}{z}-1\right)$ [16. C,

$\therefore$ Ans$=z+\dfrac{1}{z}\mp 1$

(*b*) $x^2 - y^{-2} = x^2 - (y^{-1})^2$ [$\because -2 = (-1) \times 2$. [16. C.

$= (x + y^{-1})(x - y^{-1})$

But $(x + y^{-1})(x - y^{-1})$

$= \left(x + y^{-1}\right)\left\{\left(x^{\frac{1}{2}}\right)^2 - \left(y^{-\frac{1}{2}}\right)^2\right\}$ $\because -1 = (-\frac{1}{2}) \times 2$

5. Conv.

$= \left(x + y^{-1}\right)\left(x^{\frac{1}{2}} + y^{-\frac{1}{2}}\right)\left(x^{\frac{1}{2}} - y^{-\frac{1}{2}}\right)$

$\therefore$ Ans. $= \left(x + y^{-1}\right)\left(x^{\frac{1}{2}} + y^{-\frac{1}{2}}\right)$.

3. (*a*) Suppose $x^{\frac{1}{3}} = a$, and $3y^{\frac{1}{3}} = b$.

Then the given expression, by substitution,

$= \dfrac{a+b}{a-b} + \dfrac{a^2 - ab + b^2}{a^2 + ab + b^2}$ $\because x^{\frac{2}{3}} = x^{\frac{1}{3} \times 2} = (x^{\frac{1}{3}})^2$ and $9y^{\frac{2}{3}} = (3y^{\frac{1}{3}})^2$.

$= \dfrac{(a+b)(a^2 + ab + b^2) + (a-b)(a^2 - ab + b^2)}{(a-b)(a^2 + ab + b^2)}$

But $(a+b)(a^2 + ab + b^2)$

$= a^3 + b^3 + 2ab(a+b)$ and

$(a-b)(a^2 - ab + b^2) = a^3 - b^3 - 2ab(a-b)$

[$\because$ $a^2 + ab + b^2 = a^2 - ab + b^2 + 2ab$

and $a^2 - ab + b^2 = a^2 + ab + b^2 - 2ab$]

$\therefore$ Numumerator of the above fraction

$= 2a^3 + 2ab(a+b) - 2ab(a-b) = 2a^3 + 4ab^2$.

Denominator $= a^3 - b^3$.

$\therefore$ Ans. $= \dfrac{2a^3 + 4b^2a}{a^3 - b^3}$, which by substitution

$= \dfrac{2(x^{\frac{1}{3}})^3 + 4x^{\frac{1}{3}}.9y^{\frac{2}{3}}}{(x^{\frac{1}{3}})^3 - (3y^{\frac{1}{3}})^3} = \dfrac{2x + 36x^{\frac{1}{3}}y^{\frac{2}{3}}}{x - 27y}$.

(*b*) See Exercise XII, example 3 (*a*).

Fraction $= \dfrac{x^{2n}-1}{x^n-1} - \dfrac{x^{2n}-1}{x^n+1}$ .... **1.**

But $x^{2n}-1=(x^n+1)(x^n-1)$ [16. C.

$\therefore$ **1 or Ans.** $=x^n+1-(x^n-1)=2.$ [Art. 3.

4\. (*a*) $\dfrac{\sqrt{a}}{\sqrt{a-x}}=\dfrac{1+a}{1-a}$. [Art. 22. E.

Squaring, $\dfrac{a}{a-x}=\dfrac{1+2a+a^2}{1-2a+a^2}$ [21. E, 10. A & B.

Inverting, $\dfrac{a-x}{a}=\dfrac{1-2a+a^2}{1+2a+a^2}$.

$1-\dfrac{x}{a}=1-\dfrac{4a}{(1+a)^2}$ Dividing the numerator by the Denominator.

$\therefore\ x=\dfrac{4a^2}{(1+a)^2}$..[21. C.

(*b*) $\dfrac{3}{y}+\dfrac{4}{x}=2$......**1** [Dividing the first equation by xy.

$\dfrac{4}{z}+\dfrac{3}{y}=3$......**2** [Dividing the 2nd by yz.

$\dfrac{6}{x}+\dfrac{5}{z}=4$......**3** [Dividing the 3rd by xz.

Subtracting **1** from **2**, $\dfrac{4}{z}-\dfrac{4}{x}=1$.

$\therefore\ \dfrac{6}{z}-\dfrac{6}{x}=\dfrac{3}{2}$......**4** [Multiplying both sides by $\dfrac{3}{2}$

Adding **3** and **4** $\dfrac{11}{z}=4+\dfrac{3}{2}=\dfrac{11}{2}$ $\therefore z=2.$

Substituting this value in **3** and **4**, $x=4$ and $y=3$.

5. 845 being the G. C. M. is a common factor of two required numbers.

Let $845x$ and $845y$ = the numbers required.......1

$\therefore\ 845x + 845y = 4225$

$\therefore\ x + y = 5$ [Dividing by 845.

If $x = 1$ and $y = 4$, the required numbers are 845 and 3380; and if $x = 2$ and $y = 3$, the required numbers are 1690 and 2535. [From 1.

6. Let x = No. of gallons of brandy in the mixture.

$\therefore$ cost of brandy $= 30x$ shillings.

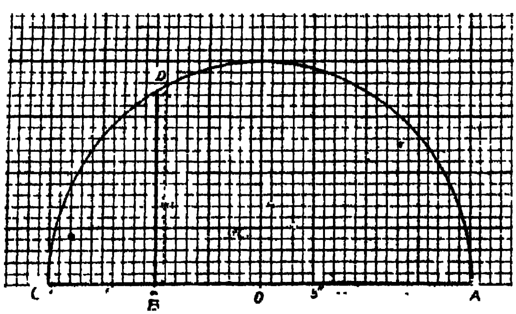
The mixture contains $(20 + x)$ gallons.

$\therefore\ 25(20 + x) = 30x$ [By the question.

$5x = 25 \times 20$; $\therefore\ x = 100$.

7. Refer to art 29.

With construction suggested therein we get from the figure $\sqrt{3} = 1{\cdot}73$.



Exercise XVI.

1. Simplify. (*a*) $\left\{ (a^{m})^{m-\frac{1}{m}} \right\}$

(*b*) $\left(a^{1+\frac{q}{p}} \right)^{\frac{p}{p+q}} + \sqrt[p]{(a^{-1})^{a^{2p}}}$

2. Prove that $(a-b)(x-a)(x-b)+(b-c)(x-b)(x-c)+$ $(c-a)(x-c)(x-a)=(a-b)(b-c)(a-c)$.

3. Reduce (*a*) $\dfrac{x(x+1)(x+2)}{3}-\dfrac{x(x+1)(2x+1)}{6}$

(*b*) $\dfrac{(n+2)^2(n+1)^2-(n-1)^2(n-2)^2}{(n+1)^3+n^3+(n-1)^3}$.

4. Resolve $(a+b+c)(ab+ac+bc)-abc$ into three factors.

5. Solve (*i*) $\dfrac{x^2+x+1}{x^2-x+1}=\dfrac{62(1+x)}{63(1-x)}$

(*ii*) $\dfrac{1}{x}+\dfrac{1}{y}=c\,;\ \dfrac{1}{y}+\dfrac{1}{z}=a\,;\ \dfrac{1}{z}+\dfrac{1}{x}=b$.

6. A banker has two kinds of money silver and gold; and a pieces of silver and b pieces of gold make up the same sum s. A person comes, and wishes to be paid the sum s with c pieces of money. How many of each must the banker give him?

7. Solve graphically $x+2y=12$ and $x-3y=2$.

1. (*a*) $\because$ By Art 5. $(a^m)^n = a^{mn}$

$\therefore \quad \{(a^m)^n\}^p = a^{mnp}.$

From this formula, the given expression

$$= a^{m\left(m-\frac{1}{m}\right)\left(\frac{1}{m+1}\right)}$$

$$= a^{m \cdot \frac{m^2-1}{m} \cdot \frac{1}{m+1}} = a^{m-1}$$

(*b*) The first term, by Art. 5,

$$= a^{\frac{p+q}{p} \times \frac{p}{p+q}} = a^1.$$

$$\text{2nd term} = \left\{\frac{a^{2p}}{(a^{-1})^{-p}}\right\}^{\frac{1}{p}} = \frac{a^{2p\times\frac{1}{p}}}{a^{-1\times-p\times\frac{1}{p}}} = \frac{a^2}{a} = a$$

$\therefore$ Ans. $= 2a$.

2. The first term $= (a-b)\ \{x^2-(a+b)x+ab\}$ [13. C.

$= (a-b)x^2-(a^2-b^2)x+ab(a-b)$

2nd term $= (b-c)\ \{x^2-(b+c)x+bc\}$

$= (b-c)x^2-(b^2-c^2)x+bc(b-c)$

3rd term $= (c-a)\ \{x^2-(c+a)x+ac\}$

$= (c-a)x^2-(c^2-a^2)x+ac(c-a).$

Adding, $\{(a-b)x^2+(b-c)x^2+(c-a)x^2\}$

$-\{(a^2-b^2)x+(b^2-c^2)x^2+(c^2-a^2)x\}$

$+ab(a-b)+bc(b-c)+ca(c-a)$

$= x^2.o - x.o+ab(a-b)+bc(b-c)+ca(c-a)$

$= ab(a-b)-a^2c+b^2c+ac^2-bc^2$

$= ab(a-b)-c(a-b)\ (a+b)+c^2(a-b)$ $\left[\because -a^2c+b^2c = -c(a^2-b^2)\right.$

$= (a-b)\ \{ab-c(a+b)+c^2\}$

$= (a-b)\ (a-c)\ (b-c)$ [13. D.

3. (*a*) $\frac{x(x+1)}{3}$ being a common factor of the two terms can be taken outside a bracket, thus :—

$$\frac{x(x+1)}{3}\left\{x+2-\frac{2x+1}{2}\right\}$$

$$=\frac{x(x+1)}{3}\left\{x+2-x-\tfrac{1}{2}\right\}$$

$$=\frac{x(x+1)}{3}\times\frac{3}{2}=\frac{x(x+1)}{2}$$

3. (*b*) Numerator $=\{(n+2)(n+1)\}^2-\{(n-1)(n-2)\}^2$

$=(n^2+3n+2)^2-(n^2-3n+2)^2$ [Art. 13. A & D.

$=(2n^2+4)\times 6n.$ [16. C.

Denominator $=(n+1)^3+(n-1)^3+n^3$

But $(n+1)^3+(n-1)^3$

$=2n\{(n+1)^2-(n+1)(n+1)+(n-1)^2\}$ [18 A.

$=2n\{2n^2+2-(n^2-1)\}=2n(n^2+3).$

$\therefore$ Denmr $=2n(n^2+3)+n^3=3n^3+6n$

$=3n(n^2+2).$

$\therefore$ Ans. $=\frac{2(n^2+2)\times 6n}{3n(n^2+2)}=4.$

4. $(ab+bc)(a+b+c)+a(a+b+c)-abc$

$=b(a+c)(a+b+c)+ac(a+b+c-b)$ [Art. 17.

$=b(a+c)(a+b+c)+ac(a+c)$

$=(a+c)\{b(a+b+c)+ac\}$ [Art. 17.

$=(a+c)\{b(a+b)+c(a+b)\}$

$=(a+c)(b+c)(a+b)$ [Art. 17.

5. (i) Multiplying the equation by $\frac{1-x}{1+x}$

$$\frac{1+x+x^2}{1-x+x^2} \cdot \frac{1-x}{1+x} = \frac{62}{63} \;;\; \frac{1-x^3}{1+x^3} = \frac{62}{63}$$

$63 - 63x^3 = 62 + 62x^3$. [18. B and A.

or $125x^3 = 1$. [19 A and B,

Extracting cube root, $5x = 1$; $\therefore$ $x = \frac{1}{5}$.

(ii) Adding the three equations,

$$2\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = a + b + c.$$

$$\therefore \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{a+b+c}{2} \; \ldots\ldots\ldots \; 1$$

Subtracting the 2nd equation from 1

$$\frac{1}{x} = \frac{a+b+c}{2} - a = \frac{b+c-a}{2} \;;\; \therefore x = \frac{2}{b+c-a}$$ [20.A

Similarly $y = \frac{2}{a-b+c}$ and $z = \frac{2}{a+b-c}$.

6. Value of one piece of silver $= \frac{s}{a}$; and that of gold

Let x = no. of silver pieces required

$\therefore$ $c - x$ = ,, ,, gold ,, ,,

$\therefore$ By the question

$$x.\frac{s}{a} + \frac{c-x}{b}.s = s.$$

Rejecting s which is a common factor,

$$\frac{x}{a}+\frac{c-x}{b}=1.$$

or $\quad bx+ca-ax=ab$; $(b-a)x=ab-ac$

$$\therefore\ x=\frac{a(b-c)}{b-a}.$$

$$\therefore\ c-x=\frac{b(c-a)}{b-a}.$$

7. For the first equation

If $x=0,\ y=6$

If $y=0,\ x=12$

Plotting these points and joining them, we get the graph of $x+2y=12$.

For the second equation

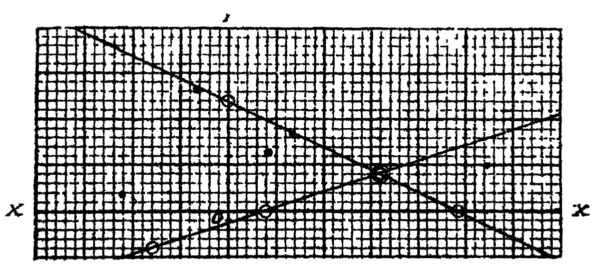
If $x=-4,\ y=-2$

If $y=0,\ x=2$

Plotting these and joining, we get the graph of

$x-3y=2$.

From the figure, the solution is $\left.\begin{array}{l}x=8\\ y=2\end{array}\right\}$ co-ordinates of the point of intersection



Exercise XVII.

1. Shew that

$$(a-b)^2(c+d)^2+4ab(c^2+d^2)-4cd(a^2+b^2)$$

is an exact square

2. (a) If $s=\frac{1}{2}(a+b+c)$ shew that

$$(s-a)(s-b)(s-c)=s^3-\frac{s}{2}(a^2+b^2+c^2)-abc.$$

(b) Show that $a(b+c)(b^2+c^2-a^2)$

$$+b(c+a)(c^2+a^2-b^2)+c(a+b)(a^2+b^2-c^2)$$

$$=2abc(a+b+c)$$

3. Simplify $\left\{\frac{a+b}{ab}\left(\frac{1}{a}-\frac{1}{b}\right)-\frac{b+c}{bc}\left(\frac{1}{c}-\frac{1}{b}\right)\right\}$

$$\div\frac{ac}{a+c}\left(\frac{1}{a}-\frac{1}{c}\right)^{-1}$$

4. Solve (i) $\dfrac{\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{2}}+b}{\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{2}}-b}=\dfrac{3}{2}$

(ii) $a(x+y)+b(x-y)=2a$
$x(a-b)-y(a+b)=-2b$

5. Find the time after h o'clock at which the hour and minute hands of a watch are distant d of the minute divisions from each other.

6. A can do a piece of work in 9 days, B in twice that time, and C can only do $\frac{3}{4}$ as much as A in a day; how long would A, B and C working together, require to do the same piece of work?

1. The given expression $=(a-b)^2(c+d)^2$

$$+4ab(c^2+2cd+d^2)-2cd\times 4ab-4cd(a^2+b^2)$$

$$=(a-b)^2(c+d)^2+4ab(c+d)^2-4cd(a^2+b^2+2ab)$$

$$=(a-b)^2(c+d)^2+4ab(c+d)^2-4cd(a+b)^2.$$

$$=\{(a-b)^2+4ab\}(c+d)^2-4cd(a+b)^2$$

$$=(a+b)^2\{(c+d)^2-4cd\} \qquad \left[\begin{aligned} &\because (a-b)^2+4ab \\ &=a^2-2ab+b^2+4ab \\ &=a^2+2ab+b^2=(a+b)^2 \end{aligned}\right.$$

$=(a+b)^2(c-d)^2$ which is manifestly a perfect square.

2. (*a*) Apply the formula of Art. 14

$$\therefore s^3-(a+b+c)s^2+(ab+ac+bc)s-abc$$

$$=s^3-(a+b+c)\cdot s^2+(2ab+2ac+2bc)\frac{s}{2}-abc$$

$$\left[\because s^2=s\times s \;\&\; s=\frac{a+b+c}{2}\right.$$

$$=s^3-\frac{s}{2}\{(a+b+c)^2-(2ab+2ac+2bc)\}-abc$$

$$=s^3-\frac{s}{2}(a^2+b^2+c^2)-abc. \qquad \text{[10 C.}$$

(*b*) $(ab+ac)b^2+(ab+ac)c^2-(ab+ac)a^2+(bc+ab)c^2$

$$+(bc+ab)a^2-(bc+ab)b^2+(ac+bc)a^2+(ac+bc)b^2$$

$$-(ac+bc)c^2.$$

$$=a^2(ac+bc+bc+ab-ab-ac)$$

$$+b^2(ab+ac-bc-ab+ac+bc)$$

$$+c^2(ab+ac+bc+ab-ac-bc)$$

$$=a^2\times 2bc+b^2\times 2ac+c^2\times 2ab$$

$$=2abc\,(a+b+c) \qquad \text{[Art 17.}$$

3. $\left(\frac{1}{a}-\frac{1}{c}\right)^{-1}=\left(\frac{c-a}{ac}\right)^{-1}=\frac{ac}{c-a}$ [Art 8.

∴ The given expression

$$\left\{\frac{(b+a)\,(b-a)}{a^2b^2}-\frac{(b+c)\,(b-c)}{b^2c^2}\right\}\div\frac{a^2c^2}{(c+a)\,(c-a)}$$

$$=\frac{(b^2-a^2)c^2-(b^2-c^2)a^2}{a^2b^2c^2}\times\frac{a^2c^2}{c^2-a^2}$$ [16. C.

$$=\frac{b^2(c^2-a^2)}{a^2b^2c^2}\times\frac{a^2c^2}{c^2-a^2}=1$$

4. (*i*) Apply Art 20. E.

$$\frac{\left(\frac{a+x}{a-x}\right)}{b}=5.\quad\therefore\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}=5b$$

Squaring $\frac{a+x}{a-x}=25b^2$

Again by 20 E., $\frac{a}{x}=\frac{25b^2+1}{25b^2-1}$

or $\frac{x}{a}=\frac{25b^2-1}{25b^2+1}$ [22. A.

$\therefore\ x=\frac{a(25b^2-1)}{25b^2+1}$ [19. C.

(*ii*) The first equation may be arranged thus :—

$x(a+b)+y(a-b)=2a$..................**1**

Multiply this by $(a+b)$ and the 2nd equation by $(a-b)$.

$\therefore\ x(a+b)^2+y(a^2-b^2)=2a(a+b)$.........**2**

$x(a-b)^2-y(a^2-b^2)=-2b(a-b)$......**3**

Adding **2** and **3**, $2x(a^2+b^2)=2(a^2+b^2)$

$\therefore\ x=1.$

Substituting this value of x in **1**,

$a+b+y(a-b)=2a$ or $y(a-b)=a-b$

$\therefore\ y=1.$

5. This is similar to Example 5, Exercise XIII.

Let x=no. of minutes past h o'clock when there is a division of d minutes between the hands.

Just at h o'clock the two hands were separate by $5h$ minutes.

But while the minute hand passes over x minutes, the hour hand travels $\frac{x}{12}$ minutes.

Now mark that the whole space travelled by the minute hand=$5h$+space travelled by the hour hand $\pm d$ (+ or − according as the minute hand goes before or after the hour hand).

$$\therefore\ x=5h+\frac{x}{12}\pm d.\quad \therefore\ \frac{11x}{12}=5h\pm d.$$

$$\therefore\ x=\frac{12}{11}(5h\pm d).$$

Note—This is a general question relating to the hands of a clock. All particular problems are dealt with by substituting suitable values for h and d. For instance, when the hands are together, $d=o$; when the hands are opposite $d=30$, when at right angles $d=15$ and h the hour mark from which the reckoning has to be made.

6. Let x=no. of days required.

In one day A can do $\frac{1}{9}$ of the work

.... B ... ... $\frac{1}{18}$ ... ..

.... C ... ... $\frac{3}{4}\times\frac{1}{9}$ ...

$$\therefore\ \frac{x}{9}+\frac{x}{18}+\frac{x}{12}=1\text{ ; hence, } x=4.$$

Exercise XVIII.

1. If $a_1+a_2+a_3+\ldots\ldots\ldots+a_n=\frac{n}{2}s.$

Prove that $(s-a_1)^2+(s-a_2)^2+\ldots\ldots+(s-a_n)^2$

$$=a_1^2+a_2^2+a_3^2+\ldots\ldots\ldots+a_n^2.$$

2. Simplify

$$\frac{a^2-(b-c)^2}{(a+c)^2-b^2}+\frac{b^2-(c-a)^2}{(a+b)^2-c^2}+\frac{c^2-(a-b)^2}{(b+c)^2-a^2}.$$

3. Prove that $\dfrac{(a+b)^3-(b+c)^3+(c+d)^3-(d+a)^3}{(a+b)^2-(b+c)^2+(c+d)^2-(d+a)^2}$

$$=\tfrac{3}{2}(a+b+c+d).$$

Solve (*i*) $\sqrt[m]{x+a}=\sqrt[2m]{x^2+5ax+b^2}.$

(*ii*) $\dfrac{5\sqrt{x+y}}{x}+\dfrac{5\sqrt{x+y}}{y}=10\tfrac{2}{3}.$

$$\frac{3\sqrt{x-y}}{y} \quad \frac{3\sqrt{x-y}}{x}=\frac{4}{9}$$

5. If $ay+bx=a,\ by-ax=b$ show that $x^2+y^2=1.$

6. A pound of tea and three pounds of sugar cost 6 shillings; but if sugar were to rise 50 per cent, and tea ten per cent, they would cost 7 shillings. Find the price of tea and sugar.

7. Solve $x^4-2x^3+x=380.$

1. $$\left.\begin{aligned}(s-a_1)^2 &= s^2-2a_1s+a_1^2\\ (s-a_2)^2 &= s^2-2a_2s+a_2^2\\ \cdots\cdots &\qquad\cdots\\ (s-a_n)^2 &= s^2-2a_ns+a_n^2\end{aligned}\right\}$$ [10. B.

Adding these, we get $s^2+s^2+\ldots$ to n terms

$-2s(a_1+a_2+\ldots$ to n terms $)$

$+a_1^2+a_2^2+\ldots$ to n terms.

$=ns^2-2s.\frac{n}{2}s+a_1^2+a_2^2+\ldots\ldots+a_n^2$ [Hyp.

$=a_1^2+a_2^2+\ldots\ldots\ldots\ldots+a_n^2$

2. $$\frac{a^2-(b-c)^2}{(a+c)^2-b^2}=\frac{(a+b-c)(a-b+c)}{(a+c+b)(a+c-b)}=\frac{a+b-c}{a+b+c}$$

$$\frac{b^2-(c-a)^2}{(a+b)^2-c^2}=\frac{(b+c-a)(b-c+a)}{(a+b+c)(a+b-c)}=\frac{b+c-a}{a+b+c}$$

$$\frac{c^2-(a-b)^2}{(b+c)^2-a^2}=\frac{(c+a-b)(c-a+b)}{(b+c+a)(b+c-a)}=\frac{c+a-b}{b+c+a}$$

Adding these we get the answer, which

$$=\frac{a+b-c+b+c-a+c+a-b}{a+b+c}=\frac{a+b+c}{a+b+c}=1$$

3. $(a+b)+(c+d)=(b+c)+(d+a)$. ..1 [By identity

Cubing both sides of 1 by 11 A,

$(a+b)^3+(c+d)^3+3(a+b)(c+d)(a+b+c+d)$

$=(b+c)^3+(d+a)^3+3(b+c)(d+a)(a+b+c+d)$

Transposing by Art. 21,

$(a+b)^3-(b+c)^3+(c+d)^3-(d+a)^3$

$=3(b+c)(d+a)(a+b+c+d)$

$-3(a+b)(c+d)(a+b+c+d)$

$=3(a+b+c+d)\{(b+c)(d+a)-(a+b)(c+d)\}$...2

Again, squaring **1**,

$$(a+b)^2+(c+d)^2+2(a+b)(c+d)$$

$$=(b+c)^2+(d+a)^2+2(b+c)(d+a)$$

Transposing $(a+b)^2-(b+c)^2+(c+d)^2-(d+a)^2$

$$=2\{(b+c)(d+a)-(a+b)(c+d)\}\ldots\ldots\ldots\mathbf{3}$$

Dividing **2** by **3**

$$\frac{(a+b)^3-(b+c)^3+(c+d)^3-(d+a)^3}{(a+b)^2-(b+c)^2+(c+d)^2-(d+a)^2}$$

$$=\tfrac{3}{2}(a+b+c+d).$$

4. (*i*) $(x+a)^{\frac{1}{m}}=(x^2+5ax+b^2)^{\frac{1}{2m}}$

Raising to the m*th* power,

$$x+a=(x^2+5ax+b^2)^{\frac{1}{2}}$$ [Art 5.

Squaring, $x^2+2ax+a^2=x^2+5ax+b^2$

or $a^2-b^2=3ax \qquad \therefore \quad x=\dfrac{a^2-b^2}{3a}$

(*ii*) 1*st* equation $\equiv 5\sqrt{x+y}\left(\dfrac{1}{y}+\dfrac{1}{x}\right)=10\tfrac{2}{3}$

2*nd* equation $\equiv 3\sqrt{x-y}\left(\dfrac{1}{y}-\dfrac{1}{x}\right)=\tfrac{1}{3}$.

$$\text{or}\ \left\{\begin{array}{l} 5\,(x+y)^{\frac{1}{2}}(x+y)=32xy \\ 3\,(x-y)^{\frac{1}{2}}(x-y)=\tfrac{1}{3}xy. \end{array}\right.$$

$$\left.\begin{array}{ll} & (x+y)^{\frac{3}{2}}=\tfrac{32}{15}xy\ldots\ldots\ldots\ldots\ldots\ldots\mathbf{1} \\ \text{or} & (x-y)^{\frac{3}{2}}=\tfrac{4}{15}xy\ldots\ldots\ldots\ldots\ldots\ldots\mathbf{2} \end{array}\right]$$ [Art. 4.

$$\therefore\ \left(\frac{x+y}{x-y}\right)^{\frac{3}{2}}=\frac{32}{4}=8.$$ [Dividing **1** by **2**

$$\therefore\quad \frac{x+y}{x-y}=8^{\frac{2}{3}}=\sqrt[3]{8^2}=4.$$

By Art 21, E. $\dfrac{x}{y}=\dfrac{5}{3}$ $\therefore$ $y=\dfrac{3x}{5}$.

Substituting this value of y in **2**

$$\left(x-\frac{3}{5}x\right)^{\frac{3}{2}}=\frac{4}{15}x.\quad \frac{3x}{5}=\frac{4x^2}{25}.$$

$$\therefore \left(\frac{2x}{5}\right)^{\frac{3}{2}}=\left(\frac{2x}{5}\right)^2$$

Dividing both sides by $\left(\dfrac{2x}{5}\right)^{\frac{3}{2}}$, $1=\left(\dfrac{2x}{5}\right)^{\frac{1}{2}}$

or $1=\left(\dfrac{2x}{5}\right)^{\frac{1}{2}}$.

$\therefore$ $\dfrac{2x}{5}=1$ or $2x=5$ or $x=\frac{5}{2}=2\frac{1}{2}$.

$\therefore$ $y=\dfrac{3x}{5}=\frac{3}{5}\times\frac{5}{2}=1\frac{1}{2}$.

5. Squaring both equations.

$a^2y^2+b^2x^2+2abxy=a^2$...... **1** [10. A.

$a^2x^2+b^2y^2-2abxy=b^2$....... **2** [10 B.

Adding **1** and **2**, $a^2y^2+b^2y^2+b^2x^2+a^2x^2=a^2+b^2$

$y^2(a^2+b^2)+x^2(b^2+a^2)=a^2+b^2$ [Art. 17.

$(a^2+b^2)(x^2+y^2)=a^2+b^2$ [Art. 17.

$\therefore$ $x^2+y^2=1$ [Dividing both sides by a^2+b^2.

6. Let x = price of 1 lb. of tea in shillings,

y = ...sugar . . . .

$\therefore$ $\dfrac{3y}{2}$ = ...on price rising

and $\dfrac{11x}{10}$ = ...tea.............

$\therefore$ By the question.

$$\left.\begin{aligned} x + 3y &= 6 \\ \frac{11x}{10} + \frac{9y}{2} &= 7 \end{aligned}\right\} \text{ or } \left\{\begin{aligned} 11x + 33y &= 66 \\ 11x + 45y &= 70 \end{aligned}\right.$$

$\therefore\ 12y = 4$ or $y = \frac{1}{3}$ $\therefore\ x = 5$ (shillings)

7. $x^4 - 2x^3 + x^2 - x^2 + x = 380$

$(x^2 - x)^2 - (x^2 - x) - 380 = 0.$

$\therefore\ x^2 - x = \frac{1 \pm \sqrt{1 + 1520}}{2}$ Art. 23. A

$= \frac{1 \pm 39}{2} = \frac{40}{2}$ or $-\frac{38}{2} = 20$ or -19.

Now $x^2 - x = 20$

$x^2 - x - 20 = 0$

$(x - 5)(x + 4) = 0$

$\therefore\ x = 5$ or -4 [23. B

$x^2 - x = -19$

$x^2 - x + 19 = 0$

$x = \frac{1 \pm \sqrt{1 - 76}}{2}$

$= \frac{1 \pm \sqrt{-75}}{2}$ [23. A.

Exercise XIX

1. If $2s = a + b + c$, prove that

$$\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s} = \frac{abc}{s(s-a)(s-b)(s-c)}$$

2. Reduce $\frac{a+b}{ab}(a^2+b^2-c^2) + \frac{b+c}{bc}(b^2+c^2-a^2$

$$+ \frac{a+c}{ac}(a^2+c^2-b^2).$$

3. (a) Find the value of

$$\frac{x+2a}{x-2a} + \frac{x+2b}{x-2b}, \text{ when } x = \frac{4ab}{a+b}.$$

(b) Divide $(a^2-bc)^3 + 8b^3c^3$ by a^2+bc.

4. Solve (i) $\sqrt{x} + \sqrt{\{a - \sqrt{ax+x^2}\}} = \sqrt{a}$.

(ii) $a\left(\frac{1}{y}+\frac{1}{z}-\frac{1}{x}\right) = b\left(\frac{1}{z}+\frac{1}{x}-\frac{1}{y}\right) = c\left(\frac{1}{x}+\frac{1}{y}-\frac{1}{z}\right) = 1.$

5. If $a+b+c=0$, prove that $a^2-bc = b^2-ca = c^2-ab$

6. A ship sails with a supply of biscuit for 60 days at a daily allowance of 1 lb. a head; after being at sea 20 days she encounters a storm, in which five men are washed overboard and damage sustained, that will cause a delay of 24 days, and it is found that each man's allowance must be reduced to $\frac{5}{7}$th of a lb. Find the original number of the crew.

1. $\dfrac{1}{s-a}+\dfrac{1}{s-b}=\dfrac{s-b+(s-a)}{(s-a)(s-b)}=\dfrac{2s-a-b}{(s-a)(s-b)}$

$=\dfrac{c}{(s-a)(s-b)}$ $\qquad$ [$\because 2s=a+b+c$.

and, $\dfrac{1}{s-c}-\dfrac{1}{s}=\dfrac{s-s+c}{s(s-c)}=\dfrac{c}{s(s-c)}$

Adding we get, $c\left\{\dfrac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right\}$

of which the Numerator $=s^2-cs+s^2-(a+b)s+ab$

$=2s^2-(a+b+c)s+ab=2s^2-2s\cdot s+ab=ab.$

$\therefore$ the given expression $=\dfrac{abc}{s(s-a)(s-b)(s-c)}$.

2. $\dfrac{a+b}{ab}=\dfrac{1}{a}+\dfrac{1}{b}$; $\dfrac{b+c}{bc}=\dfrac{1}{b}+\dfrac{1}{c}$. $\dfrac{a+c}{ac}=\dfrac{1}{a}+\dfrac{1}{c}$.

[Now see Exercise XVII, Example 2 (b) solution.]

$\because$ 1st term $=a^2\left(\dfrac{1}{a}+\dfrac{1}{b}\right)+b^2\left(\dfrac{1}{a}+\dfrac{1}{b}\right)-c^2\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$

2nd term $=b^2\left(\dfrac{1}{b}+\dfrac{1}{c}\right)+c^2\left(\dfrac{1}{b}+\dfrac{1}{c}\right)-a^2\left(\dfrac{1}{b}+\dfrac{1}{c}\right)$

3rd term $=a^2\left(\dfrac{1}{a}+\dfrac{1}{c}\right)+c^2\left(\dfrac{1}{a}+\dfrac{1}{c}\right)-b^2\left(\dfrac{1}{a}+\dfrac{1}{c}\right)$

Adding these we get $a^2\left(\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{b}-\dfrac{1}{c}+\dfrac{1}{a}+\dfrac{1}{c}\right)$

$+\,b^2\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{b}+\dfrac{1}{c}-\dfrac{1}{a}-\dfrac{1}{c}\right)$

$+\,c^2\left(\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{a}+\dfrac{1}{c}-\dfrac{1}{a}-\dfrac{1}{b}\right)$

$=a^2\times\dfrac{2}{a}+b^2\times\dfrac{2}{b}+c^2\times\dfrac{2}{c}=2(a+b+c).$

3. (a) $\dfrac{(x+2a)(x-2b)+(x+2b)(x-2a)}{(x-2a)(x-2b)}$

$$=\frac{x^2+(2a-2b)x-4ab+x^2+(2b-2a)x-4ab}{x^2-(2a+2b)x+4ab}$$

$$=\frac{2x^2+x\{2a-2b+2b-2a\}-8ab}{x^2-(2a+2b)x+4ab}$$

$$\frac{2x^2-8ab}{x^2-8ab+4ab}$$ [For, by Hyp. $x(a+b)=4ab$ or $(2a+2b)x=8ab$.

$$\frac{2(x^2-4ab)}{x^2-4ab}=2.$$

(b) Dividend $=(a^2-bc)^3+(2bc)^3$

$=(a^2-bc+2bc)\{(a^2-bc)^2-2bc(a^2-bc)+(2bc)^2\}$ 17. A.

$=(a^2+bc)\{a^4-2a^2bc+b^2c^2-2a^2bc+2b^2c^2+4b^2c^2\}$

$=(a^2+bc)\{a^4-4a^2bc+7b^2c^2\}$

$\therefore$ Quotient $=a^4-4a^2bc+7b^2c^2$.

4. (i) $\sqrt{\{a-\sqrt{ax+x^2}\}}=\sqrt{a}-\sqrt{x}$ [Transposing by 21.A.

Squaring, $a-\sqrt{ax+x^2}=a+x-2\sqrt{ax}$.

or, $-\sqrt{ax+x^2}=x-2\sqrt{ax}$.

Squaring, $ax+x^2=x^2+4ax-4x\sqrt{ax}$

or $4x\sqrt{ax}=3ax$, $4\sqrt{ax}=3a$

$4\sqrt{x}=3\sqrt{a}$. $16x=9a$. $\therefore$ $x=\dfrac{9a}{16}$.

(ii)
$$\left.\begin{aligned}
\frac{1}{y}+\frac{1}{z}-\frac{1}{x}&=\frac{1}{a}\ \ldots\ldots\ldots\ldots\ 1\\
\frac{1}{z}+\frac{1}{x}-\frac{1}{y}&=\frac{1}{b}\ \ldots\ldots\ldots\ldots\ 2\\
\frac{1}{x}+\frac{1}{y}-\frac{1}{z}&=\frac{1}{c}\ \ldots\ldots\ldots\ldots\ 3
\end{aligned}\right\}$$

Now, adding **1** and **2**, $\frac{2}{z} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$.

$\therefore\ z = \frac{2ab}{a+b}$ [22. A. & 21. C.

Similarly, adding **1** and **3**, $\frac{2}{y} \quad 1 \quad \frac{a+c}{ac}$

$\therefore\ y = \frac{2ac}{a+c}$ [22. A & 21. C.

Similarly, $x = \frac{2bc}{b+c}$ [Adding **2** & **3**.

5. $a+b = -c$ and $a+c = -b$ [21. A.

$(a+b)(a-b) = -c\,(a-b)$

$a^2 - b^2 = bc - ac$

$\therefore\ a^2 - bc = b^2 - ac \ldots\ \ldots\ \ldots$

$(a+c)(a-c) = -b(a-c)$

$\therefore\ a^2 - c^2 = bc - ab$

$\therefore\ a^2 - bc = c^2 - ab \ldots\ \ldots\ldots\ldots$

$\therefore\ a^2 - bc = b^2 - ac = c^2 - ab$ [from **1** & **2**

6. Let x = original number of the crew

$\because$ daily allowance per head = 1lb.

60 x lbs = Supply of biscuit.

In 20 days $20x$ lbs are consumed and 5 men being washed overboard the no. of crew is reduced to $x-5$.

In consequence of the damage made by the storm and of a delay of 24 days, 40 x lbs. of biscuit must be made to supply $x-5$ men for $40+24$ or 64 days, by reducing each man's allowance to $\frac{5}{7}$lbs.

$\therefore\ (x-5)\ \frac{5}{7} \times 64 = 40x \quad \therefore\ \frac{x-5}{7} \times 8 = x.$

$\therefore\ x = 40.$

Exercise XX.

1. Show that any positive fraction together with its reciprocal is greater than 2.

2. Resolve into elementary factors

$$(1-a^2)(1-b^2)(1-c^2)-(c+ab)(b+ac)(a+bc)$$

3. Simplify

(a) $$\frac{1}{(a-b)(a-c)(x+a)}+\frac{1}{(b-a)(b-c)(x+b)}+\frac{1}{(c-a)(c-b)(x+c)}.$$

(b) $$\frac{ab(x^2+y^2)+xy(a^2+b^2)}{ab(x^2-y^2)+xy(a^2-b^2)}$$

4. Solve (i) $(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}=2\sqrt{x}$.

(ii) $xyz=a^2(y+z)=b^2(x+z)=c^2(x+y)$

5. A person buys tea at 6s a lb and also some at 4s. a lb In what proportion must he mix them so that by selling his tea at 5s. 3d a lb. he may gain 20 per cent on each pound sold.

6. How many bundles of hay at Rs. 5 per thousand must a ghaswala mix with 5600 bundles at Rs. 6 per thousand in order that he may gain 20 per cent by selling the whole at 11 as. per hundred?

7. Solve $2^{2x+3}+1=32\cdot 2^x$.

1. Def. Reciprocal of a quantity is the quotient which arises from dividing unity by that quantity ; for example, x and $\frac{1}{x}$ are reciprocals : 2 and $\frac{1}{2}$ are reciprocals.

Let $\frac{x}{y}$ be any fraction. $\therefore \frac{y}{x}$ is its reciprocal [Def.

Now, $\frac{x}{y}+\frac{y}{x}$ is >2 if $\frac{x^2+y^2}{xy}>2$

or if $x^2+y^2>2xy$; but $x^2+y^2>2xy$ [Ex. XII – 1.

$\therefore \quad \frac{x}{y}+\frac{y}{x}>2$ [Dividing both sides by xy.

2. $(1-a^2)(1-b^2)(1-c^2)=1^3-(a^2+b^2+c^2)\times 1^2+(a^2b^2+a^2c^2+b^2c^2)\times 1-a^2b^2c^2$. [Art. 14.

$=1-a^2-b^2-c^2+a^2b^2+a^2c^2+b^2c^2-a^2b^2c^2 \ldots\ldots\ldots 1$

$c+ab=c(1+\frac{ab}{c})$; $b+ca=b(1+\frac{ac}{b})$ and $a+bc$

$=a(1+\frac{bc}{a}) \quad \therefore \quad (c+ab)(b+ac)(a+bc)$

$=c(1+\frac{ab}{c})\times b(1+\frac{ac}{b})\times a(1+\frac{bc}{a})$

$$=abc\left\{1^3+\left(\frac{ab}{c}+\frac{ac}{b}+\frac{bc}{a}\right)\times 1^2+\left(\frac{ab\times ac}{bc}+\frac{ab\times ac}{ac}+\frac{ac\times be}{ab}\right)\times 1+\frac{ab\times ac\times bc}{abc}\right\} \quad [14.$$

$$=abc\left\{1+\frac{ab}{c}+\frac{ac}{b}+\frac{bc}{a}+a^2+b^2+c^2+abc\right\}$$

$=abc(1+a^2+b^2+c^2)+a^2b^2c^2+a^2c^2+b^2c^2 \ldots\ldots\ldots\ldots\ldots 2$

Substracting 2 from 1 we get

$1-a^2-b^2-c^2-abc-abc(a^2+b^2+c^2)-2a^2b^2c^2$

$=1-a^2-b^2-c^2-2abc+abc-abc(a^2+b^2+c^2)-2a^2b^2c^2$.

$=(1-a^2-b^2-c^2-2abc)+abc(1-a^2-b^2-c^2-2abc)$

$=(1-a^2-b^2-c^2-2abc)(1+abc)$.

3. (*a*) See Exercise VII, Example 3 (*b*).

$$\frac{1}{(a-b)(a-c)(x+a)}-\frac{1}{(a-b)(b-c)(x+b)}+\frac{1}{(a-c)(b-c)(x+c)}$$

$$=\frac{(b-c)(x+b)(x+c)-(a-c)(x+a)(x+c)+(a-b)(x+a)(x+b)}{(a-b)(a-c)(b-c)(x+a)(x+b)(x+c)}$$

Of which the numerator by 13. A.

$$=(b-c)\{x^2+(b+c)x+bc\}-(a-c)\{x^2+(a+c)x+ac\}$$
$$+(a-b)\{x^2+(a+b)x+ab\}$$
$$=(b-c)x^2+(b^2-c^2)x+bc(b-c)-(a-c)x^2-(a^2-c^2)x$$
$$-ac(a-c)+(a-b)x^2+(a^2-b^2)x+ab(a-b)$$
$$=x^2(b-c-a+c+a-b)+x(b^2-c^2-a^2+c^2+a^2-b^2)$$
$$+bc(b-c)-ac(a-c)+ab(a-b)$$ [Exercise XVI, Example 2.
$$=ac^2-bc^2-a^2c+b^2c+ab(a-b)$$
$$=c^2(a-b)-c(a^2-b^2)+ab(a-b)$$
$$=(a-b)\{c^2-c(a+b)+ab\}$$ [Art. 17.
$$=(a-b)(a-c)(b-c).$$ [Art. 13. D.

$\therefore$ Ans. $=\dfrac{1}{(x+a)(x+b)(x+c)}$.

(*b*) Numr. $=abx^2+a^2xy+b^2xy+aby^2$ which by Art. 18.

$$=ax(bx+ay)+by(bx+ay)$$
$$=(bx+ay)(ax+by).$$

Denr. $=abx^2+a^2xy-b^2xy-aby^2$

$$=ax(bx+ay)-by(bx+ay)=(ax-by)(bx+ay)$$

Ans. $=\dfrac{ax+by}{ax-by}$.

4. (*i*) $(a+x)-(a-x)=2x$..............1 [Identity.

$\sqrt{a+x}+\sqrt{a-x}=2\sqrt{x}$ ......2 [Hyp.

$\therefore\ \sqrt{a+x}-\sqrt{a-x}=\sqrt{x}$ ....3 [Dividing 1 by 2.

$2\sqrt{a+x}=3\sqrt{x}$ [Adding 2 & 3

Squaring, $4(a+x)=9x$

or $4a=5x$

$x=\dfrac{4a}{5}$

(*ii*) $\left.\begin{aligned} a^2(y+z)&=xyz\\ b^2(x+z)&=xyz\\ c^2(x+y)&=xyz\end{aligned}\right\}$ or, $\left\{\begin{aligned} \frac{1}{xz}+\frac{1}{xy}&=\frac{1}{a^2}\\ \frac{1}{yz}+\frac{1}{xy}&=\frac{1}{b^2}\\ \frac{1}{yz}+\frac{1}{xz}&=\frac{1}{c^2}\end{aligned}\right\}$

Adding and subtracting as in Exercise XVI, Example 5 (*ii*).

$\dfrac{1}{xz}=\dfrac{1}{2}\left(\dfrac{1}{a^2}-\dfrac{1}{b^2}+\dfrac{1}{c^2}\right)=q$, Suppose,

$\dfrac{1}{xy}=\dfrac{1}{2}\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}-\dfrac{1}{c^2}\right)=r$, Suppose,

$\dfrac{1}{yz}=\dfrac{1}{2}\left(\dfrac{1}{b^2}+\dfrac{1}{c^2}-\dfrac{1}{a^2}\right)=p$, Suppose.

$\therefore$ Multiplying as in Exercise, XI Example 5 (*ii*)

$xz=\dfrac{1}{q}$; $xy=\dfrac{1}{r}$; $yz=\dfrac{1}{p}$.

$\therefore\ x=\sqrt{\dfrac{p}{qr}}$; $y=\sqrt{\dfrac{q}{rp}}$; $z=\sqrt{\dfrac{r}{pq}}$.

5. Let x = number of lbs of tea bought at 6s.

y = ... 4s.

and mixed with x lbs.

He paid for $(x+y)$ lbs of tea, $6x+4y$ Shillings.

$\therefore$ By the question

$$5\tfrac{1}{4}(x+y)=\tfrac{120}{100}\times(6x+4y).$$

$$\therefore\ 3y=13x;\quad \frac{x}{y}=\frac{3}{13}.$$

$\therefore$ Proportion $= 3 : 13$.

6. Let x = no. of bundles required.

Their cost $=\dfrac{5x}{1000}$ Rupees.

Cost of 5600 bundles @ Rs. 6 per thousand

$$=\frac{5600\times 6}{1000},\quad \text{or}\quad \frac{168}{5}\ \text{Rupees.}$$

The whole is sold for $\dfrac{(x+5600)\times\frac{11}{16}}{100}$ Rupees.

$$\frac{(x+5600)\times 11}{1600}=\frac{120}{100}\times\left(\frac{5x}{1000}+\frac{168}{5}\right)$$ [By the question.

Whence $x=2080$.

7. $2^{2x}\times 2^{8}+1=32\times 2^{x}$ [4. Conv.

$256.(2^{x})^{2}+1=32(2^{x})$ [5 Conv.

$256.(2^{x})^{2}-32(2^{x})+1=0$

$16^{2}.(2^{x})^{2}-2.16.(2^{x})+1=0$

Extracting the square root

$16.2^{x}-1=0$ [16. B and 21. E

$\therefore\ 2^{x}=\dfrac{1}{16}=\dfrac{1}{2^{4}}=2^{-4}$ [8. Conv.

$\therefore\ x=-4$.

Exercise XXI.

1. If $2s = a+b+c$, shew that $2(s-a)(s-b)(s-c)$

$$+a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b)=abc.$$

2. Resolve into elementary factors

$$x^3+2ax+(b-c)x+a^2+(b-c)a-bc.$$

3. Shew that $\dfrac{ab}{(x-a)(x-b)}+\dfrac{bc}{(x-b)(x-c)}$

$$+\frac{ac}{(x-c)(x-a)}=0$$

When $\dfrac{1}{x}=\dfrac{1}{3}\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$.

4. Solve (*i*) $\sqrt{a-m^2x}-\sqrt{b-n^2x}=\sqrt{(a+b)-(m+n)^2x}$

(*ii*) $\begin{cases} x+y+z=0 \\ (a+b)x+(a+c)y+(b+c)z=0 \\ abx+acy+bcz=1. \end{cases}$

5. A and B lay a wager of 10*s*; if A loses, he will have 25*s*. less than twice as much as B will then have; but if B loses, he will have five-seventeenths of what A will then have. How much money does each of them have?

6. Two cisterns of equal dimension are filled with water, and the taps in both are opened at the same time. If the water in one will run out in five hours, find when one cistern will have twice as much water in it as the other.

1. $2(s-a)(s-b)(s-c)+a(s-b)(s-c)$

$=(s-b)(s-c)\{2s-2a+a\}$

$=(s-b)(s-c)(b+c)\ldots\ .1$ $\qquad 2s-a=a+b+c-a$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad=b+c$ by Hyp

and $b(s-c)(s-a)+c(s-a)(s-b)$

$=(s-a)\{bs-bc+cs-bc\}=(s-a)\{s(b+c)-2bc\}$

$=s(s-a)(b+c)-2bc(s-a)\ ..2$

Adding 1 and 2,

$(b+c)\{s^2-(b+c)s+bc+s^2-as\}-2bc(s-a)$

$=(b+c)\{2s^2-(a+b+c)s\}+bc(b+c)-bc(2s-2a)$

$=(b+c)\{2s^2-2s\cdot s\}+bc\{b+c-\overline{b+c-a}\}$

$=abc.$

2. Given expression $=x^2+2ax+a^2+(b-c)(x+a)-bc$

$=(x+a)^2+b(x+a)-c(x+a)-bc$

$=(x+a)\{x+a+b\}-c\{x+a+b\}$

$=(x+a+b)(x+a-c).$

3. By Hyp. $\dfrac{1}{x}=\dfrac{1}{3}\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)=\dfrac{ab+ac+bc}{3abc}$

$\therefore\ (ab+bc+ac)x=3abc$

Now, the given expression

$=\dfrac{ab(x-c)+bc(x-a)+ac(x-b)}{(x-a)(x-b)(x-c)}.$

Of which the numerator $=abx+bcx+acx-3abc.$

$x(ab+bc+ac)-3abc=3abc-3abc=0$

$\therefore$ given expression $=0.$

4. (*i*) Squaring, $a-m^2x+b-n^2x-2\sqrt{(a-m^2x)(b-n^2x)}$

$=(a+b)-(m^2x+n^2x+2mnx)$

Cancelling, $-2\sqrt{(a-m^2x)(b-n^2x)}=-2mnx$

$\therefore\quad \sqrt{ab-an^2x-bm^2x+m^2n^2x^2}=mnx$

Squaring, $ab-x(an^2+bm^2)+m^2n^2x^2=m^2n^2x^2$

or $x=\dfrac{ab}{an^2+bm^2}$

(*ii*) Multiplying the first equation by $(b+c)$.

$(b+c)x+(b+c)y+(b+c)z=0.$

But, $(a+b)x+(a+c)y+(b+c)z=0$ [2*nd* Equation

$\therefore\quad (a-c)x+(a-b)y=0 \ldots\ldots\ldots\ldots(A)$ [By subtraction.

Again multiply the first equation by bc.

$\therefore\quad bcx+bcy+bcz=0.$

But $abx+acy+bcz=1$ [3*rd* equation

$\therefore\quad (ab-bc)x+(ac-bc)y=1$ [Subtracting the former from the latter.

or $\quad b(a-c)x+c(a-b)y=1 \ldots\ldots\ldots\ldots(B)$

again, $c(a-c)x+c(a-b)y=0 \ldots\ldots\ldots(C)$

[Multiplying (A) by (C).

and, $x\{a(b-c)-c(b-c)\}=1$ [Subtracting (C) from (B)

or $\quad x(b-c)(a-c)=1 \quad \therefore\quad x=\dfrac{1}{(a-c)(b-c)}.$

and from (A) $y=-\dfrac{(a-c)x}{a-b}$

$$=-\frac{a-c}{a-b}\times\frac{1}{(a-c)(b-c)}=-\frac{1}{(a-b)(b-c)}.$$

Substituting in $\quad z=-(x+y)$ [1st equation

we have

$$z=-\frac{1}{(a-c)(b-c)}+\frac{1}{(a-b)(b-c)}$$

$$=\frac{-(a-b)+a-c}{(a-c)(b-c)(a-b)}$$

$$=\frac{b-c}{(a-c)(b-c)(a-b)}$$

$$=\frac{1}{(a-c)(a-b)}$$

5. Let x = A's money in shillings.

y = B's money................

If A loses, he will have $x - 10$, and B, $y + 10$.

$\therefore$ By the question, $x - 10 = 2(y + 10) - 25$......1

But if B loses he will have $y - 10$ and A, $x + 10$.

$\therefore y - 10 = \frac{5}{17}(x + 10)$ 2

From 1, $x - 2y = 5$; and from 2

$5x - 17y = -220$.

Hence, $y = 35$ and $x = 2y + 5 = 75$.

6 Let x = no. of hours required.

and y=volume of water each cistern can contain.

In x hours the first cistern has discharged $\frac{xy}{5}$ and the second cistern has discharged $\frac{xy}{4}$

$\therefore$ volumes of water remaining in them are

$y - \frac{xy}{5}$ and $y - \frac{xy}{4}$ respectively.

Now, by the question, $y - \frac{xy}{5} = 2\left(y - \frac{xy}{4}\right)$

$$\therefore \quad y\left(1 - \frac{x}{5}\right) = 2y\left(1 - \frac{x}{4}\right);$$

$$\text{or} \quad 1 - \frac{x}{5} = 2 - \frac{2x}{4},$$

when $x = 3\frac{1}{3}$.

Exercise XXII.

1. Shew that $x^n - na^{n-1}x + (n-1)a^n$ is divisible by $(x-a)^2$.

2. Find the G. C. M. of

$(ax+by)^2-(a-b)(x+z)(ax+by)+(a-b)^2xz$.

and $(ax-by)^2-(a+b)(x+z)(ax-by)+(a+b)^2xz$.

3. Find the value of

$$\frac{a^n}{2na^n-2nx}+\frac{b^n}{2nb^n-2nx} \text{ when } x=\frac{a^n+b^n}{2}.$$

4. Solve (*i*) $\dfrac{\sqrt{(a+\sqrt{x})}}{\sqrt[3]{x}}+\dfrac{\sqrt{(a-\sqrt{x})}}{\sqrt[3]{x}}=\sqrt[6]{x}$

(*ii*) $7^{\frac{x}{2}+\frac{y}{3}}=2401$; $6^{\frac{x}{4}+\frac{y}{2}}=1296$.

(*iii*) $xy=ab$; $(x+y)(a-b)=(x-y)(a+b)$

5. A railway train after travelling for one hour meets with an accident which delays it one hour, after which it proceeds at $\frac{3}{5}$th of its former rate, and arrives at the terminus three hours behind time ; had the accident occurred 50 miles farther on, the train would have arrived 1 hour 20 minutes sooner : Required the length of the line.

6. If $x=cy+bz$, $y=az+cx$, $z=bx+ay$

Shew that $\dfrac{x^2}{1-a^2}=\dfrac{y^2}{1-b^2}=\dfrac{z^2}{1-c^2}$.

7. Solve $ax^n+bx^{-n}+c=0$.

1. $x^n - na^{n-1}x + na^n - a^n = x^n - a^n - na^{n-1}(x-a)$

$$= (x-a)\{x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-1}\} - (x-a)na^{n-1}.$$

This is evidently divisible by $x-a$.

Dividing this by $x-a$, the quotient

$$= x^{n-1} + ax^{n-2} + a^2x^{n-3} \ldots + a^{n-1} - na^{n-1} \ldots \quad \ldots\ldots(A)$$

But $na^{n-1} = a^{n-1} + a^{n-1} + \ldots$ &c to n terms.

$\therefore$ $(A) = (x^{n-1} - a^{n-1}) + (ax^{n-2} - aa^{n-2})$

$+ (a^2x^{n-3} - a^2a^{n-3})$ &c to n such bracketted terms.

$= (x^{n-1} - a^{n-1}) + a(x^{n-2} - a^{n-2}) + a^2(x^{n-3} - a^{n-3})$

&c to n such terms.

But each of these quantities enclosed by n brackets is divisible by $(x-a)$. [Art 20. Cor. 2.

$\therefore$ The given expression is divisible by $(x-a)(x-a)$ or $(x-a)^2$.

2. 1st quantity $= (ax + by)\{ax + by - (ax + az - bx - bz)\}$

$+ (a-b)^2xz$

$= (ax + by)b(x + y) + (ax + by)(bz - az)$

$+ (a-b)^2xz.$

$= (ax + by)b(x + y) + abxz - a^2xz + b^2yz - abyz$

$+ a^2xz - 2abxz + b^2xz.$

$= (ax + by)b(x + y) + b^2z(x + y) - abz(x + y).$

$= b(x + y)\{ax + by + bz - az\} \quad \ldots \quad \ldots \quad ..(A)$

The second quantity differs from the first only by having $-b$ instead of $+b$, and $+b$ instead of $-b$.

$\therefore$ *2nd* quantity $= -b(x + y)\{-by + ax - bz - az\}$

$= b(x + y)\{by - ax + bz + az\} \quad \ldots\ldots \quad \ldots \quad \ldots\ldots\ldots(B)$

In (A) and (B) the factors enclosed by the double bracket have no common measure greater than unity.

$\therefore$ G. C. M $= b(x+y)$.

3. $\because$ by hyp. $x = \frac{a^n + b^n}{2}$, $\therefore 2x = a^n + b^n$.

or $2nx = na^n + nb^n$.

$\therefore$ Given expression $= \frac{a^n}{na^n - nb^n} + \frac{b^n}{nb^n - na^n}$

$= \frac{a^n}{na^n - nb^n} - \frac{b^n}{na^n - nb^n} = \frac{a^n - b^n}{na^n - nb^n} = \frac{1}{n}$.

4. (*i*) $\sqrt{a+\sqrt{x}} + \sqrt{a-\sqrt{x}} = x^{\frac{1}{6}} x^{\frac{1}{3}}$ or $= x^{\frac{1}{2}}$.

[Art 4. $\because$ $\frac{1}{6}+\frac{1}{3}=\frac{1}{2}$.

Squaring, $a+\sqrt{x} + a-\sqrt{x} + 2\sqrt{a^2-x} = x$.

or $2\sqrt{a^2-x} = x - 2a$.

Again squaring, $4(a^2-x) = x^2 + 4a^2 - 4ax$

or $4x(a-1) = x^2$

$\therefore$ $x=0$ or $x=4(a-1)$

(*ii*) $2401 = 7^4$ and $1296 = 6^4$

$$\therefore \begin{cases} \frac{x}{2} + \frac{y}{3} = 4 \ldots\ldots\ldots\ldots\ldots\ldots 1 \\ \frac{x}{4} + \frac{y}{2} = 4 \ldots\ldots\ldots\ldots\ldots\ldots 2 \end{cases}$$

Again, $\frac{x}{4} + \frac{y}{6} = 2 \ldots\ldots\ldots\ldots\ldots\ldots 3$

[Multiplying 1 by $\frac{1}{2}$.

Subtracting 3 from 2

$\frac{y}{2} - \frac{y}{6}$ or $\frac{y}{3} = 2$, or $y=6$.

Substituting this value in 1, $\frac{x}{2} = 2$;

or $x=4$.

(*iii*) From the 2nd equation $\frac{x+y}{x-y}=\frac{a+b}{a-b}$.

$\therefore$ Applying 21 E. $\frac{x}{y}=\frac{a}{b}$ 1

But $xy=ab$................ 2

Multiplying 1 & 2 $x^2=a^2$

$\therefore$ $x=a$ [21. E.

Substituting this value in the first equation

$$ay=ab. \quad \therefore \quad y=b.$$

5. Let x=the length of the line in miles.

y=the rate at which train travels per hour.

$\therefore$ Time in which the train travels the line through $=\frac{x}{y}$ hours.

In one hour the train travels y miles, and being delayed one hour, it runs $x-y$ miles at the rate of $\frac{3}{5}y$ miles per hour.

$\therefore$ it runs $x-y$ miles in $\frac{(x-y)5}{3y}$ hours.

$\therefore$ By the question

$$1+1+\frac{(x-y)5}{3y}=\frac{x}{y}+3$$

$$\text{or } \frac{5}{3}\cdot\frac{x}{y}-\frac{x}{y}=1+\frac{5}{3}.$$

$$\text{or } \frac{2}{3}\cdot\frac{x}{y}=\frac{8}{3}.$$

or $x=4y$.......... ....(A)

Now to the case in which the accident occurs 50 miles further on; the train travels 50 miles at its former rate in $\frac{50}{y}$ hours, and at its reduced rate in $\frac{50\times5}{3y}$ hours.

$\therefore$ By the question

$$\frac{250}{3y}-\frac{50}{y}=1\frac{1}{3} \text{ or } \frac{100}{3y}=\frac{4}{3}.$$

$\therefore$ $y=25$; and $x=4\times25$ or 100. [From A.

6. Write the equations thus :—

$$a.\,0 + b.\,z + c.\,y = x \ldots \quad 1$$
$$a.\,z + b.\,0 + c.\,x = y \ldots \quad 2$$
$$a.\,y + b.\,x + c.\,0 = z \ldots \quad 3$$

Multiplying **1** by x and **2** by y.

$$0 + bxz + cxy = x^2$$
$$ayz + 0 + cxy = y^2$$

Subtracting, $x^2 - y^2 = bxz - ayz$

$= z(bx - ay)$ [Art. 17.

$= (bx + ay)(bx - ay)$

[$\because z = bx + ay$, 3rd equation

$\therefore \quad x^2 - y^2 = b^2x^2 - a^2y^2.$ [Art. 12.

$\therefore \quad x^2 - x^2b^2 = y^2 - a^2y^2.$

or $x^2(1 - b^2) = y^2(1 - a^2)$

$\dfrac{x^2}{1 - a^2} = \dfrac{y^2}{1 - b^2}$; similarly, from **1** and **3**

$\dfrac{x^2}{1 - a^2} = \dfrac{z^2}{1 - c^2}$

. $ax^n + \dfrac{b}{x^n} + c = 0$ [Art. 8.

$ax^{2n} + cx^n + b = 0$ [$\because x^{2n} = (x^n)^2$ Art. 5.

$\therefore \quad x^n = \dfrac{-c \pm \sqrt{c^2 - 4ab}}{2a}$ [Art. 23. A.

$\therefore \quad x = \left\{\dfrac{-c \pm \sqrt{c^2 - 4ab}}{2a}\right\}^{\frac{1}{n}}$ [Art. 21. F

Note. This equation is the general form with x raised to any power in the first term and x raised to the same negative power in the second term.

As a special case the student may try to solve $5(5^x + 5^{-x}) = 26.$

Ans := -1 or $-\frac{1}{32}$.

Exercise XXIII.

1. Prove that

$$(a+b+c)^4-(b+c)^4-(c+a)^4+a^4+b^4+c^4$$
$$=12\,abc\,(a+b+c).$$

2. Simplify (*a*) $\dfrac{1}{(x+1)(2x+1)}+\dfrac{1}{(2x+1)(3x+1)}$

$$+\frac{1}{(3x+1)(4x+1)}+\frac{1}{(4x+1)(5x+1)}$$

(*b*) $$\frac{\dfrac{a}{b}-\dfrac{b}{a}}{\dfrac{a}{b}+\dfrac{b}{a}-1}-\frac{1+\dfrac{b}{a}+\dfrac{b^2}{a^2}}{\dfrac{a}{b}-\dfrac{b^2}{a^2}}$$

3. Solve (*i*) $a^{x}\left(a^{x}+b^{-x}\right)=\dfrac{a^2b^2+1}{a^2b^2}$.

(*ii*) $xyz=a(yz-xz-xy)=b(xz-xy-yz)$
$$=c(xy-yz-xz).$$

4. If $\dfrac{a}{b}=\dfrac{c}{d}$ shew that $\dfrac{a}{b}=\dfrac{a+c}{b+d}$

5. A boy swam half a mile down a stream in 10 minutes; without the aid of the stream it would have taken him a quarter of an hour. What was the rate of the stream per hour? And how long would it take him to run against it.

6. If ax^3+bx^2+cx+d be divisible by x^2+m^2, then $ad=bc$

1. Suppose $a+b+c=s$, then $b+c=s-a$; and $c+a=s-b$.

$\therefore$ Given expression

$=s^4-(s-a)^4-(s-b)^4-(s-c)^4+a^4+b^4+c^4.$

$=s^4-(s^4-4as^3+6a^2s^2-4a^3s+a^4)-(s^4-4bs^3+6b^2s^2-4b^3s+b^4)-(s^4-4cs^3+6c^2s^2-4c^3s+c^4)+a^4+b^4+c^4$

$=s^4-3s^4+4s^3(a+b+c)-6s^2(a^2+b^2+c^2)+4s(a^3+b^3+c^3)$

[Cancelling $a^4+b^4+c^4$.

Now mark $a+b+c=s$.

And $a^3+b^3+c^3=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)+3abc$

[Art. 17 and 21. B.

$\therefore$ above result

$=s^4-3s^4+4s^3.\ s-6s^2(a^2+b^2+c^2)+4s(a+b+c)$
$\times(a^2+b^2+c^2-ab-ac-bc)+4s\ 3abc.$

$=s^4-3s^4+4s^4-6s^2(a^2+b^2+c^2)+4s^2(a^2+b^2+c^2-ab-ac-bc)+12s\times abc$

$=2s^4-2s^2\{3a^2+3b^2+3c^2-2a^2-2b^2-2c^2+2ab+2ac+2bc\}+12s\times abc$

$=2s^4-2s^2(a+b+c)^2+12s\times abc$

$=2s^4-2s^2.\ s^2+12sabc=12abc(a+b+c).$

2. (*a*) Sum of the first two quantities

$$=\frac{3x+1+x+1}{(x+1)(2x+1)(3x+1)}=\frac{2(2x+1)}{(x+1)(2x+1)(3x+1)}$$

$$=\frac{2}{(x+1)(3x+1)}\ldots\ldots\ldots\ldots\ldots\ldots\ (A)$$

Sum of the last two quantities

$$=\frac{5x+1+3x+1}{(3x+1)(4x+1)(5x+1)}=\frac{2}{(3x+1)(5x+1)}\ldots\ldots\text{(B)}$$

Adding (A) and (B),

$$2\left\{\frac{5x+1+x+1}{(x+1)(3x+1)(5x+1)}\right\}=\frac{4}{(x+1)(5x+1)}$$

(b) $$\frac{a^2-b^2}{a^2+b^2-ab}-\frac{b(a^2+ab+b^2)}{(a-b)(a^2+ab+b^2)}$$

$$\frac{a^3-ab^2-a^2b+b^3-a^2b-b^3+ab^2}{(a-b)(a^2+b^2-ab)}$$

$$=\frac{a^3-2a^2b}{(a-b)(a^2-ab+b^2)}.$$

3. (i) $$a^{-x}a^{x}+a^{-x}b^{-x}=\frac{a^2b^2}{a^2b^2}+\frac{1}{a^2b^2}.$$

$$a^{0}+(ab)^{-x}=1+\frac{1}{a^2b^2}.$$ [Art. 4.

$$1+(ab)^{-x}=1+(ab)^{-2}$$ [Art. 8 with note

$\therefore\ (ab)^{-x}=(ab)^{-2}$

or $x=2.$

(ii) Arrange thus:—

$$\left.\begin{aligned}\frac{yz-xz-xy}{xyz}&=\frac{1}{a}\\ \frac{xz-xy-yz}{xyz}&=\frac{1}{b}\\ \frac{xy-yz-xz}{xyz}&=\frac{1}{c}\end{aligned}\right\}\quad\text{or}\quad\left\{\begin{aligned}\frac{1}{x}-\frac{1}{y}-\frac{1}{z}&=\frac{1}{a}\\ \frac{1}{y}-\frac{1}{z}-\frac{1}{x}&=\frac{1}{b}.\\ \frac{1}{z}-\frac{1}{x}-\frac{1}{y}&=\frac{1}{c}\end{aligned}\right.$$

Adding 2nd and 3rd,

$$-\frac{2}{x}=\frac{1}{b}+\frac{1}{c} \qquad \therefore x=\frac{-2bc}{b+c}$$

Similarly $y=\frac{-2ac}{a+c}$; and $z=\frac{-2ab}{a+b}$

4. $\because \frac{a}{b}=\frac{c}{d} \quad \therefore ad=bc$ **1**

and $ab=ab$ .. .. **2** [By Identity.

Adding **1** and **2**, $a(b+d)=b(a+c)$

$$\therefore \frac{a}{b}=\frac{a+c}{b+d}$$

5. The boy swims half a mile without aid of the stream in 15 minutes, $\therefore$ in 10′ he swims $\frac{1}{3}$ of a mile.

Let x = velocity of the stream per hour in miles

$\therefore \frac{x}{6}$ = distance it goes in 10 minutes.

$\therefore$ By the question $\frac{x}{6}+\frac{1}{3}=\frac{1}{2}$; or $x=1$.

In one hour he can go without aid of the stream $\frac{1}{2}\times 4$ or 2 miles; but the stream impedes his progress by 1 mile in 1 hour.

$\therefore$ his real progress against the stream in one hour = 1 mile.

$\therefore$ In returning $\frac{1}{2}$ a mile he will take $\frac{1}{2}$ an hour.

6. By performing the division it is found that the quotient $=ax+b$ and the remainder $=cx-am^2x-bm^2+d$.

But the remainder, by the question, is equal to 0.

$\therefore cx=am^2x$, or $c=am^2$.**1**

and $d=bm^2$............. ....**2**

Dividing **1** by **2**

$$\frac{c}{d}=\frac{a}{b} \quad \therefore ad=bc.$$

Exercise XXIV.

1. If $x+a$ be the G. C. M. of x^2+px+q and $x^2+p'x+q'$ shew that $a=\dfrac{q-q'}{p-p'}$.

2. Simplify (a) $\dfrac{(2a-3b)^2-a^2}{4a^2-(3b+a)^2}+\dfrac{4a^2-(3b-a)^2}{9(a^2-b^2)}$

$$+\frac{9b^2-a^2}{(2a+3b)^2-a^2}.$$

(b) $\left(\dfrac{a^2+b^2}{a^2+ab+b^2}+\dfrac{a^2-b^2}{a^2-ab+b^2}\right)$

$$\div\left(\frac{a^2+b^2}{a^2+ab+b^2}-\frac{a^2-b^2}{a^2-ab+b^2}\right)$$

Solve (i) $\dfrac{3}{x-3}-\dfrac{4}{x+9}-\dfrac{5}{x-27}+\dfrac{6}{x-15}=0.$

(ii) $\sqrt{xy}=1$; $\sqrt{yz}=24x$; $\sqrt{xz}=4y$.

4. There is a certain number consisting of 2 digits. The sum of the digits is 5, and if 9 be added to the number itself the digits will be inverted. Find the number.

5. A starts from C and travels towards D, at a rate of six miles per hour: two hours afterwards B starts also from C, and going ten miles per hour reaches D four hours before A. Find the distance between C and D.

6. If $\dfrac{a}{b}=\dfrac{c}{d}$, show that $\dfrac{3a+4c}{5a+4c}=\dfrac{3b+4d}{5b+4d}$.

7. Find the condition that one root of the equation $ax^2+bx+c=0$ shall be n times the other.

1. $\because$ $x+a$ is the G. C. M. of x^2+px+q and $x^2+p'x+q'$, each of them is divisible by $x+a$; we see that there is left remainder on dividing x^2+px+q by $x+a$ which $=q-(p-a)a$. [Art. 20.

$\therefore$ $q-(p-a)a$ must$=0$ (A)

Similarly $q'-(p'-a)a=0$ (B)

$\therefore$ $q-q'-(p-p')a=0$ [Subtracting (B) from (A)

$\therefore$ $a=\dfrac{q-q'}{p-q'}$. [21 B & D

2. (*a*) Resolve the numerators and denominators into factors by Art. 16. C.

$$\frac{(2a-3b+a)\,(2a-3b-a)}{(2a+3b+a)\,(2a-3b-a)}=\frac{a-b}{a+b}\ldots\ldots\ldots\ldots\ldots\mathbf{1}$$

$$\frac{(2a+3b-a)\,(2a-3b+a)}{9(a+b)\,(a-b)}=\frac{a+3b}{3(a+b)}\ldots\ldots\ldots\ldots\mathbf{2}$$

$$\frac{(3b+a)\,(3b-a)}{(2a+3b+a)\,(2a+3b-a)}=\frac{3b-a}{3(a+b)}\ldots\ldots\ldots\ldots\mathbf{3}$$

Adding **1**, **2**, and **3**,

$$\frac{3a-3b+a+3b+3b-a}{3(a+b)}=\frac{3(a+b)}{3(a+b)}=1.$$

☞ This example is similar to Ex. XVIII, 2.

(*b*) Quantities in the 1st bracket

$$=\frac{(a^2+b^2)^2-ab(a^2+b^2)+a^4-b^4+ab(a^2-b^2)}{(a^2+ab+b^2)\,(a^2-ab+b^4)}$$

of which the Numerator $=2(a^4+a^2b^2-ab^3)$

Similarly, quantities in the 2nd bracket,

$$=\frac{2(b^4+b^2a^2-ba^3)}{(a^2+ab+b^2)\,(a^2-ab+b^2)}$$

$\therefore$ Ans. $=\dfrac{a^4+a^2b^2-ab^3}{b^4+b^2a^2-ba^3}$.

3. (*i*) Arrange thus :—

$$\frac{6}{x-15}-\frac{5}{x-27}=\frac{4}{x+9}-\frac{3}{x-3}$$

$$\frac{6x-162-5x+75}{(x-15)(x-27)}=\frac{4x-12-3x-27}{(x+9)(x-3)}.$$

or. $$\frac{x-87}{x^2-42x+405}=\frac{x-39}{x^2+6x-27}$$

$\therefore$ $$\frac{x^2-42x+405}{x-87}=\frac{x^2+6x-27}{x-39}$$ [22 A

or, Dividing the Numr. by Denr. in each

$$x+45+\frac{4320}{x-87}=x+45+\frac{1728}{x-39}.$$

or $$\frac{5}{x-87}=\frac{2}{x-39}$$ [Cancelling and dividing by 864.

$\therefore$ $5x-195=2x-174$; or $x=7$.

(*ii*) Similar to Ex. XI, 5 (*ii*) and XIV, 4. (*b*)

$xyz=96xy$ [Multiplying the equations.

$\therefore$ $z=96$.

Squaring the 1st equation, $xy=1$.

$\therefore$ $x=\frac{1}{y}$ ....(A).

$$\left(\frac{x}{y}\right)^{\frac{1}{2}}=\frac{1}{6}\,\frac{y}{x}$$ [Dividing 3rd equation by 2nd.

or, $$\left(\frac{x}{y}\right)^{\frac{3}{2}}=\frac{1}{6}$$

Substituting the value of x as given in (A),

$$\left(\frac{1}{y^2}\right)^{\frac{3}{2}}=\frac{1}{6}, \text{ or by Art. } 5, \frac{1}{y^3}=\frac{1}{6};$$

$$\text{or } y^3=6; \text{ or } y=\sqrt[3]{6}$$

$$\therefore\ x=\frac{1}{y}=\frac{1}{\sqrt[3]{6}}$$

4. Let x = ten's digit and y = unit's digit

$\therefore\quad x+y=5$.........(A).

x and y being the digits the no. $=10x+y$ [Art. 1

$\therefore$ By the question, $10x+y+9=10y+x$.

$\therefore\quad 9x-9x=-9$ or $x-y=-1$.........(B)

From (A) and (B) $x=2$ and $y=3$.

$\therefore$ No. $=23$.

5. Let x = distance required in miles.

$\therefore$ A takes $\frac{x}{6}$ hours to travel the distance and B takes $\frac{x}{10}$ hours.

But $\because$ B starts two hours later and arrives four hours earlier, $\therefore$ he performs the journey in 6 hours less than A.

$\therefore$ By the question $\frac{x}{6}-\frac{x}{10}=6$.

whence, $x=90$.

6. $\frac{a}{b} = \frac{c}{d}$. $\therefore \frac{3a}{3b} = \frac{4c}{4d}$

$\therefore$ by 22 B, $\frac{3a}{4c} = \frac{3b}{4d}$; or $\frac{3a+4c}{4c} = \frac{3b+4d}{4d}$ **1.**

[20. C.

Again, by 22 B. $\frac{a}{c} = \frac{b}{d}$

$\therefore \frac{5a}{4c} = \frac{5b}{4d}$ [Multiplying by $\frac{5}{4}$

$\frac{5a+4c}{4c} = \frac{5b+4d}{4d}$ ....**2** [20 C

$\therefore \frac{3a+4c}{5a+4c} = \frac{3b+4d}{5b+4d}$ [Dividing **1** by **2**

7. Let α be one root. Therefore $n\alpha$ is the other root [Hyp.

$\therefore$ By Art. 25, $\alpha + n\alpha = -\frac{b}{a}$ **1**

and $n\alpha^2 = \frac{c}{a}$ **2**

$\therefore$ Squaring **1**, $\alpha^2(1+n)^2 = \frac{b^2}{a^2}$ **3**

Dividing **3** by **2** $\frac{1+n^2}{n} = \frac{\frac{b^2}{a^2}}{\frac{c}{a}} = \frac{ab^2}{a^2c}$ or $= \frac{b^2}{ac}$

$\therefore ac(1+n^2) = nb^2$.

Exercise XXV.

1. If $a+b+c=0$, prove that $a^3+b^3+c^3-3abc=0$.

2. (*a*) Reduce to its lowest term

$$\frac{6x^4+10x^3+2x^2-20x-28}{3x^3+14x^2+22x+21}.$$

(*b*) Simplify $\left(\frac{x^2}{y^2}-1\right)\left(\frac{x}{x-y}-1\right)$

$$+\left(\frac{x^3}{y^3}-1\right)\left(\frac{y^2+xy}{x^2+xy+y^2}-1\right)$$

3. Solve (*i*) $(a+\sqrt{x})^{\frac{1}{3}}+(a-\sqrt{x})^{\frac{1}{3}}=b^{\frac{1}{3}}$.

(*ii*) $\begin{cases} x+y+z=a+b+c \\ bx+cy+az=cx+ay+bz=ab+bc+ca. \end{cases}$

4. It is between 2 and 3 o'clock; but a person looking at the clock, and mistaking the hour-hand for the minute-hand, fancies that the time of day is 55 minutes earlier than the reality. What is the true time?

5. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that $\frac{a+c+e}{b+d+f}=\frac{a}{b}$.

6. If $x^5-5bx+4c$ be divisible by $(x-m)^2$, then will $b^5=c^4$.

7. Show that the expression $x(x+a)(x+2a)(x+3a)+a^4$ is a perfect square.

1. $a+b+c=0 \quad \therefore \quad a+b=-c$ .....(A) [21, A.

Cubing, $a^3+b^3+3ab(a+b)=-c^3$ [11. A.

or $a^3+b^3+3ab\times(-c)=-c^3$ Substituting the value of $a+b$ as given in A.

or $a^3+b^3+c^3-3abc=0$ [By transposition

2. (a) Numr. $=6x^4-12x^2+10x^3-20x+14x^2-28$

$$=6x^2(x^2-2)+10x(x^2-2)+14(x^2-2)$$
$$=(x^2-2)(6x^2+10x+14)$$
$$=2(x^2-2)(3x^2+5x+7)$$

Denr. $=3x^3+9x^2+5x^2+15x+7x+21.$

$$=3x^2(x+3)+5x(x+3)+7(x+3)$$
$$=(x+3)(3x^2+5x+7)$$

$\therefore$ Ans. $=\dfrac{2(x^2-2)}{x+3}$.

2. (b) $$\frac{x^2-y^2}{y^2}\times\frac{x-(x-y)}{x-y}+\frac{x^3-y^3}{y^3}\times\frac{x^2+xy-(x^2+xy+y^2)}{x^2+xy+y^2}.$$

$$=\frac{(x+y)(x-y)}{y^2}\times\frac{y}{x-y}+\frac{(x^2+xy+y^2)(x-y)}{y^3}\times\frac{-y^2}{x^2+xy+y^2}$$

$$=\frac{x+y}{y}+\frac{-(x-y)}{y}=\frac{x+y-x+y}{y}=\frac{2y}{y}=2.$$

3. (i) Cubing both sides of the equation by Art. 11. A.

$$(a+\sqrt{x})+(a-\sqrt{x})+3(a+\sqrt{x})^{\frac{1}{3}}(a-\sqrt{x})^{\frac{1}{3}}\{(a+\sqrt{x})^{\frac{1}{3}}+(a-\sqrt{x})^{\frac{1}{3}}\}=b.$$

But the quantity enclosed by the double brackets $= b^{\frac{1}{3}}$ [Hyp.

$\therefore \quad 2a + 3(a^2 - x)^{\frac{1}{3}} b^{\frac{1}{3}} = b$

Transposing, $3b^{\frac{1}{3}}(a^2 - x)^{\frac{1}{3}} = b - 2a$.

Cubing, $27\,b(a^2 - x) = (b - 2a)^3$.

$27ba^2 - 27bx = b^3 - 8a^3 - 6b^2a + 12a^2b$

$\therefore \quad 27bx = 8a^3 + 15a^2b + 6ab^2 - b^3$.

$$\therefore \quad x = \frac{8a^3 + 15a^2b + 6ab^2 - b^3}{27b}.$$

(*ii*) $x + y + z = a + b + c \ldots\ldots\ldots\ldots 1$

$bx + cy + az = ab + bc + ca \ldots\ldots 2$

$cx + ay + bz = ab + bc + ac \ldots\ldots 3$

Multiply **1** by a, $\therefore \quad ax + ay + az = a^2 + ab + ac$.

From this subtract **2**

$\therefore \quad x(a - b) + y(a - c) = a^2 - bc \ldots\ldots\ldots\ldots$ (A)

Multiply **1** by b, $\therefore \quad bx + by + bz = ab + b^2 + bc$

From this subtract **3**

$x(b - c) - y(a - b) = b^2 - ac \ldots\ldots\ldots\ldots$ (B)

Multiply (A) by $(a - b)$ and (B) by $(a - c)$

$x(a - b)^2 + y(a - b)(a - c) = (a - b)a^2 - bc(a - b)$

and $x(a - c)(b - c) - y(a - b)(a - c) = (a - c)b^2 - ac(a - c)$

Adding these we get

$x\{a^2 - 2ab + b^2 + ab - ac - bc + c^2\}$

$= a\{a^2 + b^2 + c^2 - ab - ac - bc\}$

$\therefore \quad x = a$

Substituting this value in (A),

$a(a-b)+y(a-c)=a^2-bc$

or, $y(a-c)=ab-bc=b(a-c)$

$\therefore \quad y=b.$

Substituting the values of x and y in the first equation $z=c$.

4. Let x = No. of minutes past 2 o'clock, being the real time.

$\therefore$ The hour hand has advanced $\frac{x}{12}$ minutes from the 2 o'clock mark. [See Exercise XIII. 5 and Exercise XVII. 5.

But just at 2 o'clock the hour hand pointed 10 minutes.

Therefore now the hour hand points $10+\frac{x}{12}$ minutes

Hence the man mistaking one hand for another fancies the minute hand to be the hour hand.

$\therefore$ By the question.

$10+\frac{x}{12}=60-55+x$ $\quad\because$ in 60' minute hand returns to the same position

$x-\frac{x}{12}$ or $\frac{11x}{12}=5$

$\therefore \quad x=\frac{60}{11}=5\frac{5}{11}$ minutes.

5. $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=r$, suppose.

$$\left.\begin{array}{l} \because \frac{a}{b}=r, \quad \therefore a=br. \\ \because \frac{c}{d}=r. \quad \therefore c=dr \\ \because \frac{e}{f}=r \quad \therefore e=fr \end{array}\right\} \quad \therefore \frac{a+c+e}{b+d+f}=\frac{br+dr+fr}{b+d+f} = \frac{r(b+d+f)}{b+d+f}=r=\frac{a}{b}.$$

[Note. Hence, in a continued proportion the sum of the numerators divided by the sum of the denominators is equal to one of the ratios.]

6. See Exercise XXIII. Example 6. Solution.

Let the division be performed.

The quotient $= x^3 + 2mx^2 + 3m^2x + 4m^3$, and the remainder

$= x(8m^4 - 3m^4 - 5b) + 4(c - m^5)$

The latter $= 0$ by hypothesis.

$\therefore \quad x(8m^4 - 3m^4 - 5b) = 0$ as well as $4(c - m^5) = 0$.

or $m^4 = b$ and $m^5 = c$.

$\therefore \quad b^5 = m^{20} = (m^5)^4 = c^4$.

7. $\{x(x+3a)\}\{(x+2a)(x+a)\} + a^4$

$= (x^2 + 3ax)(x^2 + 3ax + a^2) + a^4$

$= (x^2 + 3ax + a^2 - a^2)(x^2 + 3ax + a^2 + a^2) + a^4$

$= (x^2 + 3ax + a^2)^2 - (a^2)^2 + a^4$ [Art 12.

$= (x^2 + 3ax + a^2)^2$ [$\because \quad (a^2)^2 = a^4$.

Which is evidently a perfect square.

Note. This is a generalisation of example 1. Exercise VII.

Exercise XXVI.

1. If $x^2 - yz = a^2$, $y^2 - xz = b^2$ and $z^2 - xy = c^2$ show that $a^2x + b^2y + c^2z - (a^2 + b^2 + c^2)(x + y + z) = 0$.

2. Simplify $\dfrac{a^4(b^2 - c^2) + b^4(c^2 - a^2) + c^4(a^2 - b^2)}{a^2(b - c) + b^2(c - a) + c^2(a - b)}$.

3. Eliminate x and y from the equations $ay + bx = bh$; $ky + hx = b^2$; and $x^2 + y^2 = b^2$.

4. Solve $\dfrac{x^3+ax^2-bx+c}{x^3-ax^2+bx+c}=\dfrac{x^2+ax-b}{x^2-ax+b}$

5. A and B run a mile. First A gives B a start of 44 yards and beat him by 51 seconds; at the second beat A gives B a start of 1 minute 15 seconds and is beaten by 88 yards. Find the time in which A and B can run a mile separately

6. If $\dfrac{x_1}{x_2}=\dfrac{x_2}{x_3}=\dfrac{x_3}{x_4} \quad \dfrac{x_n}{x_{n+1}}$

Prove that $\left(\dfrac{x_1+x_2+x_3+\ldots\ldots+x_n}{x_2+x_3+x_4+\ldots\ldots+x_{n+1}}\right)^n=\dfrac{x}{x_{n+1}}$.

1. $\left.\begin{array}{l} x^2-yz=a^2 \\ y^2-xz=b^2 \\ z^2-yx=c^2 \end{array}\right\}$ $\therefore$ by addition,
$x^2+y^2+z^2-xz-yz-xy$
$=a^2+b^2+c^2 \ldots\ldots\ldots\ldots\ldots(A)$.

Multiplying the 3 equations by x, y and z respectively,
$a^2x=x^3-xyz$; $b^2y=y^3-xyz$; and $c^2z=z^3-xyz$.

Adding, $a^2x+b^2y+c^2z=x^3+y^3+z^3-3xyz$. (B)

Again, $(a^2+b^2+c^2)(x+y+z)=(x^2+y^2+z^2-xy-xz-yz)$
$\times(x+y+z)$ [Multiplying (A) by $x+y+z$.
$=x^3+y^3+z^3-3xyz \ldots\ldots\ldots(C)$ [Art. 17.

Subtracting (C) from (B) we get
$a^2x+b^2y+c^2z-(a^2+b^2+c^2)(x+y+z)=0$

2. Denr. $=a^2(b-c)+bc(b-c)-a(b^2-c^2)$
$=(b-c)\{a^2+bc-a(b+c)\}$
$=(b-c)(a-b)(a-c) \ldots\ldots\ldots(A)$ [13. C.

Now mark for every a and a^2 in the denominator there are a^2 and a^4 respectively in the numerator ; so for every b and b^2, c and c^2 in the denominator there are b^2, b^4, c^2 and c^4 respectively in the numerator.

$\therefore$ Numerator $=(b^2-c^2)(a^2-b^2)(a^2-c^2)$.........(B)

Dividing (B) by (A) we get the answer,

which $=(a+b)(b+c)(c+a)$ [16. C.

3. Multiply the 1st equation by h and the 2nd by b.

$$\therefore \begin{cases} ahy+bhx=bh^2 \\ bky+bhx=b^3 \\ \therefore\ y(ah-bk)=b(h^2-b^2) \end{cases}$$ [Substracting the latter from the former.]

$\therefore\ y^2(ah-bk)^2=b^2(h^2-b^2)^2$............(A) [Squaring.

Again, multiply the 1st equation by k and the second equation by a.

$$\therefore \begin{cases} aky+bkx=bkh. \\ aky+ahx=ab^2 \end{cases}$$

$\therefore\ x(ah-bk)=b(ab-kh)$ [Substracting the former from the latter.]

$\therefore\ x^2(ah-bk)^2=b^2(ab-kh)^2$.........(B) [Squaring.

Adding (A) and (B),

$(x^2+y^2)(ah-bk)^2=b^2\{(h^2-b^2)^2+(ab-hk)^2\}$

But $x^2+y^2=b^2$ [3rd equation

$\therefore\ (ah-bk)^2=(h^2-b^2)^2+(ab-kh)^2.$

or $a^2h^2+b^2k^2-2abhk=(h^2-b^2)^2+a^2b^2+k^2h^2-2abhk.$

or $a^2(h^2-b^2)-k^2(h^2-b^2)=(h^2-b^2)^2.$

or $a^2-k^2=h^2-b^2$; or $a^2+b^2=h^2+k^2.$

4 By 22E, $\dfrac{x^3+c}{ax^2-bx}=\dfrac{x^2}{ax-b}$

$\dfrac{x^3+c}{x(ax-b)}=\dfrac{x^2}{ax-b}$. [Art. 17.

$\dfrac{(ax-b)x}{x^3+c}=\dfrac{ax-b}{x^2}$ [22. A.

$\therefore\ ax-b=0$ [See Exercise XIV.—4 (a). Solution.

or $ax=b$ or $x=\dfrac{b}{a}$ [21. D.

5. Let x=number of seconds in which A can run a mile.

y=number of seconds in which B can run a mile.

First, A gives B a start of 44 yards

or $\dfrac{y\times 44}{1760}=\dfrac{y}{40}$ seconds' start.

$\therefore\ x+\dfrac{y}{40}=y-51$. or $x-\dfrac{39y}{40}=-51$.......1.

Secondly, A gives B a start of 75 seconds and is beaten by 88 yards

or in $\dfrac{x\times 88}{1760}=\dfrac{x}{20}$ seconds.

$\therefore\ 75+x-\dfrac{x}{20}=y$; or $\dfrac{19x}{20}-y=-75$ ....2.

From 1 and 2 $\begin{cases} x=300 \text{ seconds or 5 minutes} \\ y=360 \text{ seconds or 6 minutes.} \end{cases}$

6. $\dfrac{x_1+x_2+x_3+\dots+x_n}{x_2+x_3+x_4+\dots+x_{n+1}}=\dfrac{x_1}{x_2}$ Applying note Ex. XXV. 5.

$$\therefore\ \left(\frac{x_1+x_2+x_3+\dots+x_n}{x_2+x_3+x_4+\dots+x_{n+1}}\right)^n=\left(\frac{x_1}{x_2}\right)^n$$

$$=\frac{x_1}{x_2}\times\frac{x_2}{x_3}\times\frac{x_3}{x_4}\times\dots\times\frac{x_n}{x_{n+1}}\text{ to } n \text{ factors}$$

$$=\frac{x_1}{x_{n+1}}$$ [For if $a_1=a_2=a_3=\dots=a_n$; $a_1^n=a_1\times a_2\times\dots\times a_n$

Exercise XXVII

1. (*a*) Simplify $(x^m)^{\frac{1}{m}}-(x^{1+\frac{1}{m}})^{\frac{m}{m+1}}+\sqrt[2]{x^{2m}}$.

(*b*) Divide $x+y+z-3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}$.

2. If $xy+yz+xz=1$, shew that

$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

3. Solve (*i*) $2^x : 2=8 : 1$

(*ii*) $\begin{cases} xy=c(x+y+z)\\ yz=a(x+y+z)\\ xz=b(x+y+z)\end{cases}$

4. It is between 11 and a quarter before 12 o'clock ; and it is observed that the number of minute spaces between the hands measured in the direction in which they move is to the number 10 minutes afterwards, as 3 is to 2. Find the time indicated by the clock.

5. Eliminate x and y form the equations

$x+y=a$, $x^2+y^2=b^2$, $x^3+y^3=c^3$.

6. If $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$

Shew that

$$(b-c)x+(c-a)y+(a-b)z=0.$$

1. (*a*) $(x^m)^{\frac{1}{m}}=x^1$ and $\left(x^{1+\frac{1}{m}}\right)^{\frac{m}{m+1}}=x^{\frac{m+1}{m}\times\frac{m}{m+1}}$
$=x$ [Art 5.

By art 7. $\sqrt[m]{x^{2m}}=x^{\frac{2m}{m}}=x^2$

$\therefore$ Ans$=x^2$.

(*b*) $\because$ $x=(x^{\frac{1}{3}})^3$ [Art 5. Conv.

$\therefore$ Supposing $x^{\frac{1}{3}}=a$, $y^{\frac{1}{3}}=b$, $z^{\frac{1}{3}}=c$,

We make the dividend $=a^3+b^3+c^3-3abc$.

$=(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$

[See solution, Exercise XIV 3.

Also with the same supposition the divisor $=a+b+c$.

$\therefore$ Ans. $=a^2+b^2+c^2-ab-ac-bc$

$$=x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}-x^{\frac{1}{3}}z^{\frac{1}{3}}-y^{\frac{1}{3}}z^{\frac{1}{3}}.$$

2. $\because$ $xy+yz+xz=1$............(A) Hyp.

$$\left.\begin{aligned}x(y+z)&=1-yz\\ y(x+z)&=1-xz\\ z(x+y)&=1-xy\end{aligned}\right\}\text{.....................(B)}$$

Now the given expression by 13 C.

$$=\frac{x\{1-(y^2+z^2)+y^2z^2\}+y\{1-(x^2+z^2)+x^2z^2\}+z\{1-(x^2+y^2)+x^2y^2\}}{(1-x^2)(1-y^2)(1-z^2)}$$

Of which the numerator—

$$x - x(y^2+z^2) + xy^2z^2$$
$$+y - y(x^2+z^2) + x^2z^2y$$
$$+z - z(x^2+y^2) + zx^2y^2.$$
$$=(x+y+z) - x^2(y+z) - y^2(x+z) - z^2(x+y)$$
$$+xyz(xy+xz+yz)$$
$$=(x+y+z) - x(1-yz) - y(1-xz) - z(1-xy) + xyz$$

[By (A) and (B).

$=xyz+xyz+xyz+xyz=4xyz.$ [Cancelling x, y, and z.

∴ The given expression

$$=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$$

3. (*i*) $\frac{2^x}{2}=8$ or $2^{x-1}=2^3$. [Art. 6.

∴ $x-1=3$; or $x=4$.

3. (*ii*) Divide the 1st equation by the 2nd. and by the third separately,

$$\left.\begin{array}{l} \dfrac{x}{z}=\dfrac{c}{a} \\[2ex] \dfrac{y}{z}=\dfrac{c}{b} \end{array}\right\} \text{ or } \left\{\begin{array}{l} x=\dfrac{cz}{a} \\[2ex] y=\dfrac{cz}{b} \end{array}\right.$$

Substitute these values of x and y in the first equation,

$$\frac{cz}{a}\times\frac{cz}{b}=c\left(\frac{cz}{a}+\frac{cz}{b}+z\right)$$

or $$\frac{c^2z^2}{ab}=cz\left(\frac{bc+ac+ab}{ab}\right)$$

Dividing by $\dfrac{cz}{ab}$, $cz=bc+ac+ab.$

$$\therefore\quad z=\frac{bc+ac+ab}{c}$$

$$\therefore\quad x=\frac{c}{a}\left(\frac{bc+ac+ab}{c}\right)=\frac{bc+ac+ab}{a}.$$

Similarly, $y=\dfrac{bc+ac+ab}{b}$

4. Let x=No. of minutes after 11 O'clock as it is indicated by the clock.

Now in x minutes the hour hand travels $\frac{x}{12}$ minutes. [XIII—5. XVII—5 XXV—4.

Just at 11 the two hands were apart by 5×11 or 55 minutes.

$\therefore$ Division between the hands now

$$=55+\frac{x}{12}-x \text{ or } \frac{660-11x}{12} \ldots\ldots\ldots\ldots\text{(A)}$$

But 10 minutes afterwards, the division between the hands $=55+\frac{x+10}{12}-(x+10)$ or $\frac{550-11x}{12}$ ....(B)

According to the question, divide (A) by (B).

$\therefore \frac{660-11x}{550-11x}=\frac{5}{2}$ $\therefore \frac{110-2x}{10}=5$ [22.E.

$\therefore x=30.$

$\therefore$ Time required=half past 11.

5. Squaring the 1st equation. $x^2+y^2+2xy=a^2$.

But $x^2+y^2=b^2$. [2nd equation.

$\therefore b^2+2xy=a^2$; or $2xy=a^2-b^2$. [21. A.

$$\therefore xy=\frac{a^2-b^2}{2}\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

Again cube the first equation, [11. A.

$$x^3+y^3+3xy(x+y)=a^3\ldots\ldots\ldots\text{(B)}$$

But $x^3+y^3=c^3$ [3rd equation.

and $3(x+y)=3a$ [1st equation.

$\therefore$ Substituting these in (B)

$$c^3+\frac{a^2-b^2}{2}\times 3a=a^3$$

or $\quad 2c^3+3a^3-3ab^2=2a^3$

or $\quad 2c^3=3ab^2-a^3$.

6. Apply note Solution 5. Exercise XXV

$\therefore$ each fraction $=\dfrac{x+y}{b+c-a+c+a-b}$; or

$$=\frac{x+z}{b+c-a+a+b-c}\,; \quad \text{or} \quad =\frac{y+z}{c+a-b+a+b-c}$$

$$\therefore \frac{x+y}{\;}=\frac{x+z}{\;}=\frac{y+z}{\;}$$

Now from $\dfrac{x+y}{c}=\dfrac{x+z}{b}$ we get $bx+by-cx-cz=0$

Similarly from $\dfrac{y+z}{a}=\dfrac{x+y}{c}$, $\quad cy+cz-ax-ay=0$

and from $\dfrac{x+z}{b}=\dfrac{y+z}{a}$, $\quad ax+az-by-bz=0$

Adding these three together

$$x(b-c)+y(c-a)+z(a-b)=0.$$

Exercise XXVIII.

1. (*a*) The greatest common measure of two numbers is 555, and the numbers are as 3 : 5. What are the numbers?

(*b*) If $a+b+c+ \ldots\ldots$ to n terms $=s$,

Show that $\frac{s-a}{s}+\frac{s-b}{s}+\frac{s-c}{s}+\ldots\ldots$ to n terms $=n-1$.

2. Prove that $\left(\frac{b}{c}+\frac{c}{b}\right)^2+\left(\frac{c}{a}+\frac{a}{c}\right)^2+\left(\frac{a}{b}+\frac{b}{a}\right)^2$

$$=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{b}{a}\right)$$

3 Solve (*i*) $\frac{x^2+2x+2}{x+1}+\frac{x^2+8x+20}{x+4}$

$$=\frac{x^2+4x+6}{x+2}+\frac{x^2+6x+12}{x+3}$$

(*ii*) $\frac{ax+by}{cz}=\frac{cz+ax}{by}=\frac{by+cz}{ax}=x+y+z.$

4. A person buys apples at 6*d*. a dozen and $2\frac{1}{2}$ times as many pears at 4d. a dozen: after mixing them he sells them at 5*d* a dozen, gaining 4*s*. on the whole. How many dozen of each does he buy?

5. Eliminate y and z from the equations $x+y+z=p$; $xy+xz+yz=q$; and $xyz=r$.

6. Two numbers end with the same digit, and being divided by 7 the quotient of each is the remainder of the other: the sum of these also is 7. What are the numbers?

Solve $\left(\frac{x}{x+1}\right)^2+\left(\frac{x}{x-1}\right)^2=\frac{45}{16}.$

1. (a) [Somewhat similar to Ex. XV – 5.

$\because$ 555 is the G. C. M.

Let $555x$ and $555y$ be the two numbers, x and y having no common factor greater than unity

By the question

$$\frac{555x}{555y} = \frac{3}{5}\ ; \text{ or } \frac{x}{y} = \frac{3}{5}.$$

$\therefore$ If $x=3, y=5$,

the numbers are 555×3, and 555×5, or 1665 and 2775.

(b) $$\frac{s-a}{s} + \frac{s-b}{s} + \frac{s-c}{s} + \ldots\ldots \text{to } n \text{ terms}$$

$$= \left(\frac{s}{s} - \frac{a}{s}\right) + \left(\frac{s}{s} - \frac{b}{s}\right) + \left(\frac{s}{s} - \frac{c}{s}\right) + \ldots\ldots$$

$$= \left(1 - \frac{a}{s}\right) + \left(1 - \frac{b}{s}\right) + \left(1 - \frac{c}{s}\right) + \ldots\ldots\ldots\ldots$$

$$= (1+1+1+\&c \text{ .to } n \text{ terms}) - \frac{1}{s}(a+b+c+\text{to } n \text{ terms})$$

$$= n - \frac{1}{s} \times s \quad [\text{For by hyp. } a+b+c+\ldots\text{to } n \text{ terms} = s$$

$$= n - 1$$

2. $$\left(\frac{b}{c}+\frac{c}{b}\right)^2 + \left(\frac{c}{a}+\frac{a}{c}\right)^2 + \left(\frac{a}{b}+\frac{b}{a}\right)^2$$

$$= \left(\frac{b^2}{c^2}+\frac{c^2}{b^2}+2\right) + \left(\frac{c^2}{a^2}+\frac{a^2}{c^2}+2\right) + \left(\frac{a}{b}+\frac{b}{a}\right)^2 \qquad [10. \text{ A.}$$

$$= 4 + \left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right) + \left(\frac{c^2}{a^2}+\frac{a^2}{c^2}\right) + \left(\frac{a}{b}+\frac{b}{a}\right)^2.$$

$$= 4 + \frac{1}{c^2}(a^2+b^2) + c^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right) + \left(\frac{a}{b}+\frac{b}{a}\right)^2.$$

$$= 4 + \frac{1}{c^2}(a^2+b^2) + c^2 . \frac{a^2+b^2}{a^2b^2} + \frac{(a^2+b^2)^2}{a^2b^2}$$

$$= 4 + (a^2+b^2)\left\{\frac{1}{c^2} + \frac{c^2}{a^2b^2} + \frac{a^2+b^2}{a^2b^2}\right\} \ \ldots\ldots\ldots\text{A.}$$

But the quantity within the double bracket

$=\frac{a^2b^2+c^4+(a^2+b^2)c^2}{a^2b^2c^2}$ of which the numerator by 13 A.

$=(c^2+a^2)(c^2+b^2)$.

$\therefore$ (A) $=3+\frac{(a^2+b^2)(a^2+c^2)(b^2+c^2)}{a^2b^2c^2}$

$=4+\frac{a^2+b^2}{ab}\cdot\frac{a^2+c^2}{ac}\cdot\frac{b^2+c^2}{bc}$.

$=4+\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{a}{c}+\frac{c}{a}\right)\left(\frac{b}{c}+\frac{c}{b}\right)$.

3. (*i*) Dividing the numerator by the denominator,

$\left(x+1+\frac{1}{x+1}\right)+\left(x+4+\frac{4}{x+4}\right)$

$=\left(x+2+\frac{2}{x+2}\right)+\left(x+3+\frac{3}{x+3}\right)$

or $\frac{1}{x+1}-\frac{2}{x+2}=\frac{3}{x+3}-\frac{4}{x+4}$ [21. A.

or $\frac{-x}{x^2+3x+2}=\frac{-x}{x^2+7x+12}$

This gives one value of x as 0. [see solution 4 (*a*) of Ex. XIV.

Also $x^2+7x+12=x^2+3x+2$ or $4x=-10$, or $x=-2\frac{1}{2}$.

(*ii*) Apply the theorem, established in Ex. XXV.—5. Note, to the first three equations

$\therefore\ \frac{(ax+by)+(cz+ax)+(by+cz)}{cz+by+ax}=x+y+z$

or $\frac{2(ax+by+cz)}{ax+by+cz}=x+y+z$

or $x+y+z=2$... (A)

$\therefore\ \frac{ax+by}{cz}=2$.

By Art 22 C. $\frac{ax+by+cz}{cz}=\frac{2+1}{1}$ or 3 1

Similarly from the second equation,

$$\frac{ax+by+cz}{by}=3\ldots\ldots\ldots 2$$

So, also from the third equation,

$$\frac{ax+by+cz}{cz}=3\ldots\ldots 3 \quad \therefore \ 1,\ 2, \text{ and } 3 \text{ are equal}$$

$$\therefore \quad ax=by=cz\ ; \quad \text{or } y=\frac{ax}{b} \text{ and } z=\frac{ax}{c}$$

Substituting these values in (A)

$$x+\frac{ax}{b}+\frac{ax}{c}=2.$$

$$\text{or } x\,(bc+ac+ab)=2bc.$$

$$\therefore \quad x=\frac{2bc}{bc+ac+ab}.$$

$$\therefore \quad y=\frac{ax}{b}=\frac{2ac}{bc+ac+ab} \quad \text{and}$$

$$z=\frac{ax}{c}=\frac{2ab}{bc+ac+ab}.$$

4. Let $x=$ No. of dozen of apples

and $\frac{5}{2}x=$ ....pears.

He paid for apples $\frac{x}{2}$ shillings and for pears $\frac{5}{2}x.\frac{1}{3}$ or $\frac{5}{6}x$ s.

$$\therefore \quad \text{Total cost}=\frac{x}{2}+\frac{5x}{6}=\frac{8x}{6}. \text{ shillings}$$

and he sold $x+\frac{5}{2}x$ dozen for

$$\frac{7x}{2}\times\frac{5}{12} \text{ shillings or } \frac{35x}{24}\ s.$$

$$\therefore \quad \text{By the question, } \frac{35x}{24}-\frac{8x}{6}=4.$$

$$\therefore \frac{3x}{24}=4 \qquad \therefore \ x=32 \text{ and } \tfrac{5}{2}x=80.$$

5 From the first equation $y+z=p-x$. (A)

and from the 3rd, $yz=\frac{r}{x}$. ... .(B)

Now, from the 2nd equation,

$yz+x(y+z)=q$.

$\therefore \frac{r}{x}+x(p-x)=q$ Substituting (A) and (B).

or, $r+px^2-x^3=qx$

$\therefore x^3-px^2+qx-r=0$.

6. Let m, n and p, n be the digits of the two numbers, and let q, r the quotient and remainder respectively obtained by dividing the first number by 7.

$\therefore$ By the question

$\frac{10m+n}{7}=q+\frac{r}{7}$... .. 1 [Art. 1.

$\frac{10p+n}{7}=r+\frac{q}{7}$. 2

$q+r=7$.. 3

Subtracting 2 from 1 and multiplying both sides by 7.

$10(m-p)=6(q-r)$

$\therefore \frac{m-p}{q-r}=\frac{3}{5}$. Now let $q-r=5$ $\therefore$ $m-p=3$.

Again $\because$ $q-r=5$ and $q+r=7$ [3rd equation.

$\therefore$ $q=6$ and $r=1$.

Substituting these values in the first equation.

$10m+n=7q+r=43$
the numbers required.
Similarly $10p+n=7r+q=13$.

7. $\left(\frac{x}{x+1}+\frac{x}{x-1}\right)^2 - 2\cdot\frac{x}{x+1}\cdot\frac{x}{x-1} = \frac{45}{16}$. $\left[\because\ a^2+b^2 = (a+b)^2-2ab.\right.$

$$\left(\frac{2x^2}{x^2-1}\right)^2 - \frac{2x^2}{x^2-1} - \frac{45}{16} = 0$$

$$\therefore \quad \frac{2x^2}{x^2-1} = \frac{1 \pm \sqrt{1+\frac{45}{4}}}{2} = \frac{1 \pm \frac{7}{2}}{2} \qquad \text{| Art. 23.}$$

$$= \frac{9}{4} \text{ or } -\frac{5}{4}$$

$$\frac{2x^2}{x^2-1} = \frac{9}{4}.$$

$$8x^2 = 9x^2 - 9$$

$$x^2 = 9$$

$$\therefore \quad x = \pm 3$$

$$\frac{2x^2}{x^2-1} = -\frac{5}{4}.$$

$$8x^2 = 5x^2 + 5$$

$$3x^2 = 5$$

$$x^2 = \frac{5}{3}$$

$$\therefore \quad x = \pm\sqrt{\frac{5}{3}}$$

Exercise XXIX.

1. If $x+y+z=2a$, and $x^2+xy+y^2+a^2=2a(x+y)$,

Show that

$$(x-a)^2+(y-a)^2+(z-a)^2=a^2.$$

2. Simplify $\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}$

3. Solve (i) $\frac{2x+3y-4z}{x+5} = \frac{3x+4y-2z}{5x}$

$$= \frac{4x+2y-3z}{4x-1} = \frac{x+y-z}{6}$$

$$(ii) \begin{cases} (x+y)(y+z)=35 \\ (x+z)(y+z)=42 \\ (x+y)(x+z)=30. \end{cases}$$

4. In a certain lake the tip of a bud of lotus was seen a span above the surface of the water. Forced by the wind it gradually advanced and was submerged at a distance of two cubits. What is the depth of the water ?

5. (*a*) Eliminate x, y and z from the equations

$$x^2(y+z) = a^3, \quad y^2(x+z)=b^3, \quad z^2(x+y)=c^3$$

and $(x+y)\ (x+z)\ (y+z) = abc$.

(*b*) If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$; show that

$$\left\{\frac{a^n+c^n+e^n}{b^n+d^n+f^n}\right\}^{\frac{1}{n}}=\frac{a}{b}.$$

6. A boat goes up stream 30 miles and down stream 44 miles in 10 hours Again it goes up stream 40 miles and down stream 55 miles in 13 hours. Find the rates of the stream and the boat.

7. Solve $x\ (x+a)\ (x+2a)\ (x+3a)+a^4=0$.

1. From the first hypothesis $z=2a-(x+y)$

Squaring, $z^2=4a^2+(x+y)^2-4a(x+y)$

$=4a^2+x^2+2xy+y^2-4a(x+y)$ **1**

But $\because\ 2a(x+y)=x^2+xy+y^2+a^2$ [Hyp.

$\therefore\ 4a(x+y)=2x^2+2xy+2y^2+2a^2$

$\therefore$ From **1**, $z^2=4a^2+x^2+2xy+y^2-2x^2-2xy-2y^2-2a^2$

$=2a^2-x^2-y^2$. ..

Now, $(x-a)^2+(y-a)^2+(z-a)^2$

$=x^2+y^2+z^2-2a(x+y+z)+3a^2$

$=x^2+y^2+z^2-2a\times 2a+3a^2$ [$\because$ By Hyp. $x+y+z=2a$

$=x^2+y^2+(2a^2-x^2-y^2)-4a^2+3a^2$

[Substituting **2** for z^2.

2. See Exercise III, Example 3 (b) Solution, also Exercise XX, Example 3 (a) Solution.

The given expression $= \dfrac{(x-b)(x-c)}{(a-b)(a-c)}$

$$-\frac{(x-c)(x-a)}{(a-b)(b-c)}+\frac{(x-a)(x-b)}{(a-c)(b-c)}$$

$$=\frac{(x-b)(x-c)(b-c)-(x-c)(x-a)(a-c)+(x-a)(x-b)(a-b)}{(a-b)(b-c)(a-c)}$$

Of which the numerator $=(a-b)(b-c)(a-c)$

[Exercise XVI—2

$\therefore$ Ans.$=1$.

3. (i) Apply the theorem established in Exercise XXV, Example 5. Solution Note. See also Exercise XXVIII—3 (ii).

$\therefore$ $\dfrac{\text{Sum of the numerators of the three equations}}{\text{Sum of the denominators of the three equations}}$ $=$ One given equation.

$\therefore$ $\dfrac{10(x+y-z)}{10(x+1)}$ or $\dfrac{x+y-z}{x+1}=\dfrac{x+y-z}{6}$.

$\therefore$ $x+1=6$. or $x=5$.

Substituting this in the first three equations, we get $y=4$. and $z=2$.

(ii) Multiply the three equations together as in Exercise XI, Example 5 (ii) Solution.

$\therefore$ $(x+y)^2\times(x+z)^2\times(y+z)^2=5\times7\times7\times6\times5\times6$

$\therefore$ $(x+y)(x+z)(y+z)=5\times7\times6$ ....(A).

Divide (A) by the three equations separately,

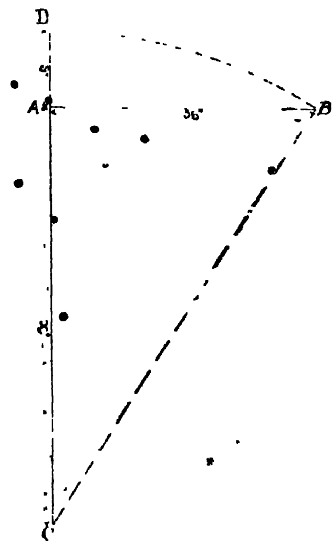
$\therefore$ $x+y=5$, $x+z=6$; $y+z=7$...... . ...(B)

$\therefore$ $x+y+z=9$ [Adding as in Ex. XVI.—4 (ii)

Subtracting from this the three new equations marked (B)

$x=2$, $y=3$; and $z=4$.

4. Let a straight line AB represent the surface of the lake, and let AC drawn below at right angles to AB represent the depth of water, C being the bottom. Produce CA to D making AD = a span or 9 inches;



and let the lotus forced by the wind advance and be submerged at B.

$\therefore$ CD = CB. And AB = two cubits, or 36 inches.

Suppose x = depth of water or length AC.

$\therefore$ By Pythagoras's theorem [Euc. I. 47 or Hall and Steven's school Geometry II. 29.]

$$CA^2 + AB^2 = CB^2.$$

or $x^2 + 36^2 = (x+9)^2 = x^2 + 18x + 81.$

$\therefore$ $18x = 1296 - 81 = 1215.$

$\therefore$ $x = 67\frac{1}{2}$ inches or 5ft. $7\frac{1}{2}$ inches.

5. (*a*) Multiplying the first three equation.

$$x^2y^2z^2(y+z)(x+z)(x+y)=a^3b^3c^3.$$

But $(x+y)(z+x)(y+z)=abc.$

$\therefore \quad x^2y^2z^2=a^2b^2c^2$ or $xyz=abc.$

$\therefore \quad 2xyz=2abc$..................................(A)

Add the three equations, and (A)

$$\therefore \quad x^2(y+z)+y^2(z+x)+z^2(x+y)+2xyz$$
$$=a^3+b^3+c^3+2abc.$$

But $x^2(y+z)+y^2(z+x)+z^2(x+y)+2xyz$

$=(x+y)(y+z)(z+x)$ [Ex VII 2 (*b*)

$\therefore \quad (x+y)(x+z)(y+z)=a^3+b^3+c^3+2abc.$

But $(x+y)(x+z)(y+z)=abc$

$\therefore \quad abc=a^3+b^3+c^3+2abc$;

or $a^3+b^3+c^3+abc=0.$

(*b*) $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}.$

$$\therefore \quad \frac{a^n}{b^n}=\frac{c^n}{d^n}=\frac{e^n}{f^n}$$

$$\therefore \quad \frac{a^n+c^n+e^n}{b^n+d^n+f^n}=\frac{a^n}{b^n}$$ See note Exercise XXV Solution 5.

$$\therefore \quad \left(\frac{a^n+c^n+e^n}{b^n+d^n+f^n}\right)^{\frac{1}{n}}=\frac{a}{b}$$ [Extracting *nth* root.

6. Let x and y be the rates of the boat and the stream respectively in miles per hour. In going up stream 30 miles, the boat takes $\dfrac{30}{x-y}$ hours; and in going 44 miles downstream, the boat takes $\dfrac{44}{x+y}$ hours.

$\therefore$ By the question $\dfrac{30}{x-y}+\dfrac{44}{x+y}=10$...1.

Similarly $\frac{40}{x-y}+\frac{55}{x+y}=13$ .. **2**

Multiplying **1** by 4 and **2** by 3.

$\frac{120}{x-y}+\frac{176}{x+y}=40$; and $\frac{120}{x-y}+\frac{165}{x+y}=39$;

Subtracting the 2nd from the 1st $\frac{11}{x+y}=1$

or $x+y=11$ .. **3**

Similarly multiplying **1** by 5 and **2** by 4.

$\frac{150}{x-y}+\frac{220}{x+y}=50$; and $\frac{160}{x-y}+\frac{220}{x+y}=52$.

Subtracting the 1st from the second $\frac{10}{x-y}=2$,

or $x-y=5$. **4**

$\therefore$ From **3** and **4**

$x=8$, $y=3$.

7. Referring to example 7 of Exercise XXV solution we have

$$x^2+3ax+a^2=0.$$

$$\therefore\ x=\frac{-3a\pm\sqrt{9a^2-4a^2}}{2}$$ [By Art. 23

$$=\frac{-3a\pm\sqrt{5a^2}}{2}.$$

$$=\frac{a\{-3\pm\sqrt{5}\}}{2}.$$

Note—Particular cases of this general equation may be formed by giving particular numerical values to *a*.

Exercise XXX.

1. If a, b, c be three unequal positive integers, prove that $(a+b+c)(ab+ca+bc) > 9\,abc$.

2. Simplify $\dfrac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}$.

3. Solve (i) $\dfrac{(5x^4+10x^2+1)(5a^4+10a^2+1)}{(x^4+10x^2+5)(a^4+10a^2+5)}=ax$.

(ii) $\begin{cases} x^2+a^2+y^2+b^2=\sqrt{2}\{x(a+y)-b(a-y)\} \\ x^2-a^2-y^2+b^2=\sqrt{2}\{x(a-y)+b(a+y)\} \end{cases}$

4. Eliminate x, y, z from the equations.

(a) $\begin{cases} x-z=a(x-y) \\ \dfrac{1}{x}-\dfrac{1}{z}=b\left(\dfrac{1}{x}-\dfrac{1}{y}\right) \\ x^2z=y^3. \end{cases}$

(b) $\begin{cases} \dfrac{x}{y}+\dfrac{y}{z}+\dfrac{z}{x}=a \\ \dfrac{x}{z}+\dfrac{y}{x}+\dfrac{z}{y}=b. \\ \left(\dfrac{x}{y}+\dfrac{y}{z}\right)\left(\dfrac{y}{z}+\dfrac{z}{x}\right)\left(\dfrac{z}{x}+\dfrac{x}{y}\right)=c. \end{cases}$

5. If $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=\dfrac{1}{a+b+c}$; shew that

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{2n+1}=\frac{1}{a^{2n+1}+b^{2n+1}+c^{2n+1}}.$$

6. Two passengers have together 5 cwt. of luggage and are charged for the excess above the weight allowed 5*s.* 2*d* and 9*s* 10*d.* respectively; but if the luggage had all belonged to

one of them he would have been charged 19*s.* 2*d.* How much luggage is each passenger allowed to carry free of charge and how much luggage had each passenger?

7. Construct a graph which will enable you to convert, at sight, degrees Fahrenheit into degrees Centigrade, and *vice versa.*

1. $(a+b+c)(ab+ac+bc) > 9abc$

if $a^2b+a^2c+abc+ab^2+abc+b^2c+abc+ac^2+bc^2 > 9abc$.

or if $ab^2+ac^2+bc^2+a^2b+a^2c+b^2c > 6abc$.

or if $(ab^2-2abc+ac^2)+(bc^2-2abc+a^2b)$

$+(a^2c-2abc+b^2c) > 0$.

or if $a(b-c)^2+b(c-a)^2+c(a-b)^2 > 0$.

But $a(b-c)^2+b(c-a)^2+c(a-b)^2 > 0$

$\because$ every square number is > 0 [Ex. XII. 1 Solution

and $\because$ a, b, c are unequal positive integers [Hypothesis.

$\therefore$ $(a+b+c)(ab+ac+bc) > 9abc$

2. Denominator $= a^3-b^3-3ab(a-b)+b^3-c^3$

$-3bc(b-c)+c^3-a^3-3ac(c-a)$

[$\because$ $(a-b)^3 = a^3-b^3-3ab(a-b)$

$= 3ac(a-c)-3bc(b-c)-3ab(a-b)$

$= 3\{c(a^2-b^2)-c^2(a-b)-ab(a-b)\}$

$= 3(a-b)\{c(a+b)-c^2-ab\}$

$= -3(a-b)\{c^2-c(a+b)+ab\}$

$= -3(a-b)(c-a)(c-b).$ [Art. 13. C.

Now mark, the numerator differs from the denominator in having a^2 instead of a, b^2 instead of b, and c^2 instead of c.

$\therefore$ Numerator $= -3(a^2-b^2)(c^2-a^2)(c^2-b^2)$

$\therefore$ Ans. $=(a+b)(a+c)(b+c)$ [16. C.

3 (*i*) Multiply both sides by $\dfrac{a^4+10a^2+5}{x(5a^4+10a^2+1)}$

$$\therefore \quad \frac{5x^4+10x^2+1}{x^5+10x^3+5x}=\frac{a^5+10a^3+5a}{5a^4+10a^2+1}.$$

$$\therefore \quad \frac{x^5+10x^3+5x}{5x^4+10x^2+1}=\frac{5a^4+10a^2+1}{a^5+10a^3+5a}$$ [22. A.

$$\frac{x^5+5x^4+10x^3+10x^2+5x+1}{x^5-5x^4+10x^3-10x^2+5x-1}$$

$$=\frac{1+5a+10a^2+10a^3+5a^4+a^5}{1-5a+10a^2-10a^3+5a^4-a^5}.$$ [22 F.

$\left(\dfrac{x+1}{x-1}\right)^5=\left(\dfrac{1+a}{1-a}\right)^5$; or $\dfrac{x+1}{x-1}=\dfrac{1+a}{1-a}$ [Extracting 5th root

$\therefore \; x=\dfrac{1}{a}$ [Art. 22 E.

(*ii*) $x^2+b^2=\sqrt{2}\,(ax+by)$..........(A) [Adding the two equations

and $y^2+a^2=\sqrt{2}\,(xy-ab)$......(B) [Subtracting 2nd from 1st.

Multiplying (A) and (B)

$(x^2+b^2)(y^2+a^2)=2(ax+by)(xy-ab)$.........(C)

But $(x^2+b^2)(y^2+a^2)=a^2x^2+b^2y^2+x^2y^2+a^2b^2$

$=(a^2x^2+b^2y^2+2abxy)+(x^2y^2+a^2b^2-2abxy)$

$=(ax+by)^2+(xy-ab)^2$ [16.B and A.

Substituting this value in C,

$$(ax+by)^2+(xy-ab)^2-2(ax+by)(xy-ab)=0$$

$\therefore\quad \{(ax+by)-(xy-ab)\}^2=0$ [16. B.

$\therefore\quad ax+by-(xy-ab)=0$; or $ax+by=xy-ab$.

$$\therefore\quad y=\frac{a(x+b)}{(x-b)}$$

Substituting this value of y in (A),

$$x^2+b^2=\sqrt{2}\,a\left\{x+\frac{bx+b^2}{x-b}\right\}$$

$$=\sqrt{2}.\ \frac{a(x^2+b^2)}{x-b}$$

$\therefore\quad 1=\sqrt{2}.\ \frac{a}{x-b}$, [Dividing by x^2+b^2.

or, $x-b=a\sqrt{2}$

or $x=b+a\sqrt{2}$. [21. B.

$$\therefore\quad y=\frac{a(x+b)}{x-b}=a+b\sqrt{2}$$

4. (*a*) From the 2nd equation we get

$$\frac{z-x}{xz}=\frac{b(y-x)}{xy}$$

or $x-z=\frac{bz(x-y)}{y}$. But $x-z=a(x-y)$ 1st Equation.

$\therefore\quad a(x-y)=\frac{bz(x-y)}{y}$, or $a=\frac{bz}{y}$

$\therefore\quad \frac{a}{b}=\frac{z}{y}$ (A)

From the third equation

$\frac{x^2}{y^2}=\frac{y}{z}=\frac{b}{a}$ [From A and Art. 22. A.

$\frac{x}{y}=\frac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}}$ (B)

Now, from the 1st equation. $x - ax = z - ay$

or $x(1-a) = z - ay$.

$\therefore \quad \dfrac{x}{y}(1-a) = \dfrac{z}{y} - a$ [Dividing by y

Substituting the value of $\dfrac{x}{y}$ as given in (B) and $\dfrac{z}{y}$ as given

in (A), $\dfrac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}}(1-a) = \dfrac{a}{b} - a = \dfrac{a(1-b)}{b}$.

$\therefore \quad b^{\frac{3}{2}}(1-a) = a^{\frac{3}{2}}(1-b)$

or $a^{\frac{3}{2}} - b^{\frac{3}{2}} = ab(a^{\frac{1}{2}} - b^{\frac{1}{2}})$

(*b*) From the 1*st* equation

$$\frac{x}{y} + \frac{y}{z} = a - \frac{z}{x}; \quad \frac{y}{z} + \frac{z}{x} = a - \frac{x}{y};$$

and $\dfrac{z}{x} + \dfrac{x}{y} = a - \dfrac{y}{z}$.

Multiplying these together,

$$\left(\frac{x}{y} + \frac{y}{z}\right)\left(\frac{y}{z} + \frac{z}{x}\right)\left(\frac{z}{x} + \frac{x}{y}\right)$$

$$= a^3 - \left(\frac{z}{x} + \frac{x}{y} + \frac{y}{z}\right)a^2$$

$$+ a\left(\frac{z}{x}\cdot\frac{x}{y} + \frac{z}{x}\cdot\frac{y}{z} + \frac{x}{y}\cdot\frac{y}{z}\right) - \frac{z}{x}\cdot\frac{x}{y}\cdot\frac{y}{z}$$

[Art 14.

$= a^3 - a.\,a^2 + ba - 1$ [From 1*st* and 2*nd* equations

But

$$\left(\frac{x}{y} + \frac{y}{z}\right)\left(\frac{y}{z} + \frac{z}{x}\right)\left(\frac{z}{x} + \frac{x}{y}\right) = c$$ [3rd equation.

$\therefore \quad a^3 - a^3 + ab - 1 = c$

or $ab - c - 1 = 0$

5. By 21. A. $\frac{1}{a}+\frac{1}{b}=\frac{1}{a+b+c}-\frac{1}{c}$.

or $\frac{a+b}{ab}=\frac{-(a+b)}{c(a+b+c)}$.

$\therefore \quad a+b=0$ [See solution 4 (*a*) Ex. XIV.

$\therefore \quad a=-b$.

$\therefore \quad a^{2n+1}=(-b)^{2n+1}=-b^{2n+1}$. [$\because$ $2n+1$ is an odd no. See solution 1. Ex. V.

Now $\left(\frac{1}{a+b+c}\right)^{2n+1}=\frac{1}{c^{2n+1}}$ [$\because$ $a+b=0$.

$=\frac{1}{b^{2n+1}-b^{2n+1}+c^{2n+1}}=\frac{1}{b^{2n+1}+a^{2n+1}+c^{2n+1}}$ [$\because$ $-b^{2n+1}=a^{2n+1}$.

6. Let x = No. of lbs. each passenger is allowed to carry free of charge, and y = weight of one's luggage in lbs.

$\therefore$ The other's luggage weighs $5\times 112-y$ [$\because$ 1 cwt $=112$ lbs

The two passangers together pay 15 shillings.

But if the luggage had belonged to one person, he would have had to pay $19\frac{1}{6}$ *s*.

$\therefore$ the charge for x lbs $=19\frac{1}{6}-15$ or $4\frac{1}{6}$ *s*.

$\therefore$ . . . . 1lb $=\frac{4\frac{1}{6}}{x}=\frac{25}{6x}$ *s*.

Now, one passenger pays for $y-x$ lbs.

and the other for $560-y-x$ lbs.

$\therefore$ By the question

$$(y-x)\,\frac{25}{6x}=5\tfrac{1}{6} \quad \cdots\cdots\cdots 1$$

$$\left(560-y-x\right)\times\frac{25}{6x}=9\tfrac{5}{6}$$

Adding and subtracting,

$$(560-2x)\frac{25}{6x}=15 \quad \cdots\cdots\cdots \quad (A)$$

$$(560-2y)\frac{25}{6x}=\frac{14}{3} \quad \cdots\cdots\cdots \quad (B)$$

Dividing (A) by (B) $\dfrac{560-2x}{560-2y}=\dfrac{45}{14}$.

or $90y-28x=560\times 31 \quad \cdots (C)$

Again from (A), $\dfrac{560\times 25}{6x}-\dfrac{50x}{6x}=15$.

or $\dfrac{7000}{3x}=15+\dfrac{25}{3}=23\tfrac{1}{3}=\dfrac{70}{3}$

$\therefore \quad x=100$

Substituting this value in (C)

$y=224$ lbs or 2 cwt.

$\therefore \quad 5\times 112-y\text{ lbs}=5\text{ cwt}-2\text{ cwt}=3\text{ cwt}.$

7 Let x° in the Centigrade scale be the same temperature as y° in the Fahrenheit scale.

In the Centigrade scale, the freezing point stands at 0°, and in the Fahrenheit at 32°.

In the Centigrade scale, the boiling point is at 100°; and in the Fahrenheit at 212°.

$$\therefore \quad \frac{x}{100}=\frac{y-32}{212-32}.$$

whence $9x=5y-160$.

Drawing the graph of this equation, the abscissa will give us the temperature in Centigrade scale, whilst the correspond-

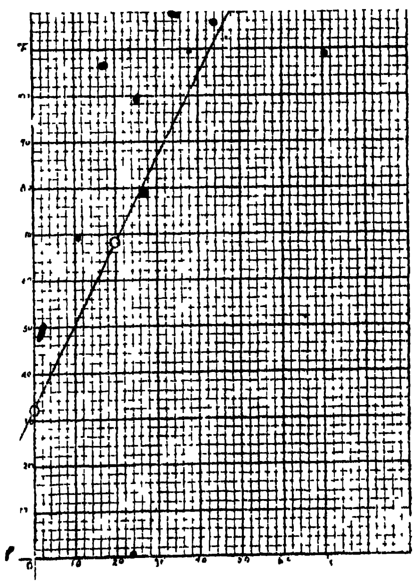
ing ordinates will give the corresponding temperature in the Fahrenheit scale.

For the graph we have

if $x=0$, $y=32$ [In the above equation

and if $x=20$, $y=68$ [any value of x would do, but 20 is very suitable and convenient in type.

Joining these points we have the graph required.



FINIS.